20.01.20

Entire function

Theorem:- If $g(z)$ is an entire function then $f(z) = e^{g(z)}$ is also entire and $f(z) \neq 0$. Conversely, if $f(z)$ is an entire function which is never zero then $f(z)$ must be of the form $e^{g(z)}$

Proof:- Suppose $f(z)$ is an entire function with no zeros so that $f(z) \neq 0$

Then the function $\phi(z) = \frac{f'(z)}{f(z)}$ is also an entire function.

Thus we have,

$$\int_{z_0}^{z} \phi(z) = \int_{z_0}^{z} \frac{f'(z)}{f(z)} dz$$

$$\Rightarrow \int_{z_0}^{z} \phi(z) = \Big[\log f(z)\Big]_{z_0}^{z}$$

$$\Rightarrow \int_{z_0}^{z} \phi(z) = \log f(z) - \log f(z_0)$$

$$\Rightarrow \log f(z) = \log f(z_0) + \int_{z_0}^{z} \phi(z)\, dz$$

$$\Rightarrow \log f(z) = a + \phi_1(z) \quad , \text{ where } a = \log f(z_0) \text{ and } \phi_1(z) = \int_{z_0}^{z} \phi(z)\, dz$$

$$\Rightarrow \log f(z) = g(z), \text{ (say)}$$

$$\Rightarrow f(z) = e^{g(z)}$$

where $g(z)$ is entire, since $\phi(z)$ is entire implies $\phi_1(z)$ is entire and so $a + \phi_1(z)$ is entire.

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Note :- e^z has no zeros.

Primary factors

For $z \in \mathbb{C}$, $E(z,0) = 1-z$ and
for $p = 1, 2, 3, \ldots$, $E(z,p) = (1-z)\exp\left\{z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^p}{p}\right\}$

Note :- $E(z,p)$ (i.e. primary factor) will always have 1 as its one zero.

These functions introduced by Weierstrass are called Weierstrass primary factors.

Each primary factor has one simple zero at $z=1$

Weierstrass Factorization Theorem

Statement :- Let $f(z)$ be an entire function and let $\{z_n\}$ be the sequence of non-zero zeros of f whose sole limiting point is the point at infinity repeated according to multiplicity. Suppose f has a zero at $z=0$ of order $m \geq 0$. Then there is an entire function $g(z)$ and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n\right)$$

Proof: Assume that the numbers $z_1, z_2, \ldots, z_n, \ldots$ are non-zero zeros of f.

Construct Weierstrass primary factors.

$$E(z,0) = 1-z$$

$$E(z,p) = (1-z)\exp\left\{z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots+\frac{z^p}{p}\right\} \quad \text{---(1)}$$

Here each primary factor has one simple zero at $z=1$

Taking log both sides in eq^n(1),

$$\log E(z,p) = \log(1-z) + \log\exp\left\{z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots+\frac{z^p}{p}\right\}$$

$$= \log(1-z) + z + \frac{z^2}{2}+\frac{z^3}{3}+\cdots+\frac{z^p}{p}$$

$$= \left\{-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots-\frac{z^p}{p}-\frac{z^{p+1}}{p+1}-\cdots\right\} + \left\{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right\}$$

$$= -\frac{z^{p+1}}{p+1}-\frac{z^{p+2}}{p+2}-\cdots$$

$$= -\sum_{n=1}^{\infty}\frac{z^{p+n}}{p+n}$$

$$\Rightarrow \left|\log E(z,p)\right| = \left|\sum_{n=1}^{\infty}\frac{z^{p+n}}{p+n}\right|$$

$$\leq \sum_{n=1}^{\infty}\frac{|z^{p+n}|}{|p+n|}$$

$$\leq |z^p|\sum_{n=1}^{\infty}\frac{|z|^n}{p+n}$$

$$\leq |z|^p\left\{\frac{|z|}{p+1}+\frac{|z|^2}{p+2}+\cdots\right\}$$

$$\leq |z|^{p+1}\left\{\frac{1}{p+1}+\frac{|z|}{p+2}+\frac{|z|^2}{p+3}+\cdots\right\}$$

$$\Rightarrow \left|\log E(z,p)\right| \le |z|^{p+1}\left\{\frac{1}{p+1}+\frac{|z|}{p+1}+\frac{|z|^2}{p+1}+\cdots\right\}$$

$$< \frac{|z|^{p+1}}{(1-|z|)} \quad \text{--- (2)} \quad \left\{\because \frac{1}{p+1}<1 \text{ and } (1+|z|+|z|^2+\cdots) = (1-|z|)^{-1}\right\}$$

Now, if $k>1$ and $|z| \le \frac{1}{k}$

then $-|z| \ge \frac{-1}{k}$

$$\Rightarrow 1-|z| \ge 1-\frac{1}{k} = \frac{k-1}{k}$$

$$\Rightarrow \frac{1}{1-|z|} \le \frac{k}{k-1} \quad \text{--- (3)}$$

From (2) and (3),

$$\left|\log E(z,p)\right| \le |z|^{p+1}\left(\frac{k}{k-1}\right) \quad \text{--- (4)}$$

Now, let $r_n = |z_n|$, $n = 1, 2, \ldots$

Then r_n increases with n, so there exist a sequence $\{p_n\}$ of positive integers such that the series $\sum_{n=1}^{\infty}\left(\frac{R}{r_n}\right)^{p_n+1}$ is convergent $\forall R>0$ --- (5)

Now, let $f(z) = z^m e^{g(z)} \prod E\left(\frac{z}{z_n}, p_n\right)$ --- (6)

where $g(z)$ is an entire function.

The entire function $f(z)$ given by (6) is found to have the required property according to the specifications of the theorem.

To prove this, we observe that if $|z| \leq R$ and $|z_n| > 2R$ then $\left|\frac{z}{z_n}\right| = \frac{|z|}{|z_n|} < \frac{R}{2R} = \frac{1}{2}$

$$\Rightarrow \frac{R}{z_n} < \frac{1}{2} \quad \text{for } |z| = R$$

Thus the series $\sum_{n=1}^{\infty} \left(\frac{R}{r^n}\right)^{p_n+1}$ is convergent.

Now, taking $k=2$ in (4),

$$|\log E(z,p)| \leq |z|^{p+1}\left(\frac{2}{2-1}\right)$$

$$\Rightarrow |\log E(z,p)| \leq 2|z|^{p+1}$$

$$\Rightarrow \left|\log E\left(\frac{z}{z_n}, p_n\right)\right| \leq 2\left|\frac{z}{z_n}\right|^{p_n+1}$$

$$\Rightarrow \left|\log E\left(\frac{z}{z_n}, p_n\right)\right| \leq 2\left(\frac{R}{r_n}\right)^{p_n+1} = M_n$$

Here, $\sum_{n=1}^{\infty} M_n = \sum 2\left(\frac{R}{r_n}\right)^{p_n+1}$

Therefore, $\sum_{n=1}^{\infty} M_n$ is convergent

Hence by Weierstrass M-test, the series $\sum_{n=1}^{\infty} \log E\left(\frac{z}{z_n}, p_n\right)$ is uniformly convergent.

Since the series $\sum \log(1+a_n)$ and the product $\prod(1+a_n)$ both are in identical nature

Therefore, the product $\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n\right)$ is also uniformly convergent —(7)

If in addition to the points z_n, $z=0$ is also a zero of order m,

Therefore we have to introduce the factor z^m in the product (7) and then the infinite product $z^m \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n\right)$ defines an entire function whose only zeros are $0, z_1, z_2, z_3, \cdots$

i.e., $\phi(z) = z^m \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n\right)$ is an entire function —— (A)

Since $f(z)$ and $\phi(z)$ are entire functions, therefore $\frac{f(z)}{\phi(z)}$ is also an entire function.

$\Rightarrow e^{g(z)}$ is an entire function (from (6) & (A))

$\Rightarrow f(z) = e^{g(z)} \phi(z)$ is also an entire function

Hence $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n\right)$ is an entire function

Theorem:- Every function which is meromorphic in the whole complex plane is the quotient of two entire functions.

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Imp Q.1. If $|z| < \frac{1}{2}$, then (i) $|E(z,p)-1| \leq 2e|z|^{p+1}$

(ii) $|E(z,p)| \leq \exp(2|z|^{p+1})$

H.W. (iii) $|E(z,p)| \geq \exp\left(-\frac{2}{|z|^{p+1}}\right)$

Proof:- (i) Given that :- $|z| < \frac{1}{2}$

To prove that :- $|E(z,p)-1| \leq 2e|z|^{p+1}$

If $p=0$ then,

$$|E(z,0)-1| = |(1-z)-1|$$
$$= |z|$$

$$\Rightarrow \quad |E(z,0)-1| \leq 2e|z|$$

Now, since

$$1-z = \exp(\log(1-z))$$
$$= \exp\left[-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots\right]$$
$$= \exp\left[-\sum_{n=1}^{\infty}\frac{z^n}{n}\right] \quad \text{—— (1)}$$

Therefore, for $p \neq 0$, we have

$$E(z,p) = (1-z)\exp\left\{z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots+\frac{z^p}{p}\right\}$$
$$= (1-z)\exp\left\{\sum_{n=1}^{p}\frac{z^n}{n}\right\}$$
$$= \exp\left[-\sum_{n=1}^{\infty}\frac{z^n}{n}\right]\exp\left[\sum_{n=1}^{p}\frac{z^n}{n}\right] \quad \{\text{from (1)}\}$$
$$= \exp\left\{-\sum_{n=1}^{\infty}\frac{z^n}{n}+\sum_{n=1}^{p}\frac{z^n}{n}\right\}$$

$$E(z,p) = \exp\left\{-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right\} \quad \text{--- (2)}$$

Hence,

$$|E(z,p)-1| = \left|\exp\left\{-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right\} - 1\right|$$

$$\leq \exp\left|\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right| - 1 \quad \text{--- (3)}$$

$\left\{ \because \text{For a complex number } a \text{ (say)} \quad |e^a - 1| \leq e^{|a|} - 1 \right\}$

Now,

$$\left|\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right| \leq \sum_{n=p+1}^{\infty} \frac{|z|^n}{n}$$

$$= \frac{|z|^{p+1}}{p+1} + \frac{|z|^{p+2}}{p+2} + \cdots$$

$$\leq |z|^{p+1} + |z|^{p+2} + \cdots \quad \left\{\because \frac{1}{p+1} < 1, \frac{1}{p+2} < 1 \ldots\right\}$$

$$= |z|^{p+1}\left\{1 + |z| + |z|^2 + \cdots\right\}$$

$$= |z|^{p+1} \sum_{n=0}^{\infty} |z|^n$$

$$< |z|^{p+1} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= |z|^{p+1} \left(\frac{1}{1 - 1/2}\right)$$

$$= 2|z|^{p+1} \quad \text{--- (4)}$$

From (3) and (4), we get

$$|E(z,p)-1| \leq \exp\{2|z|^{p+1}\} - 1$$

$$\leq 2|z|^{p+1} \exp\left(2|z|^{p+1}\right)$$

$$\left\{\because e^{x}-1 \leq x e^{x}\right\}$$

$$\leq 2|z|^{p+1} \exp\left(2\left|\frac{1}{2}\right|^{p+1}\right)$$

$$\leq 2|z|^{p+1} \exp(1) \qquad \left\{\because 2\left|\frac{1}{2}\right|^{p+1} < 1\right\}$$

$$\boxed{|E(z,p)-1| \leq 2e|z|^{p+1}}$$

(ii) We have,

$$(1-z) = \exp(\log(1-z)) = \exp\left[-\sum_{n=1}^{\infty} \frac{z^n}{n}\right]$$

$$\therefore\ E(z,p) = \cancel{ex}\,(1-z)\exp\left\{\sum_{n=1}^{p} \frac{z^n}{n}\right\}$$

$$= \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n}\right\} \exp\left\{\sum_{n=1}^{p} \frac{z^n}{n}\right\}$$

$$= \exp\left\{-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right\}$$

Now,

$$|E(z,p)-1| \geq |E(z,p)| - 1$$

$$\Rightarrow |E(z,p)| - 1 \leq |E(z,p)-1|$$

$$\leq \left|\exp\left\{-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right\} - 1\right|$$

$$\leq \exp\left|\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right| - 1$$

$$\Rightarrow |E(z,p)| \leq \exp\left|\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right|$$

$$\Rightarrow |E(z,p)| \leq \exp\left(2|z|^{p+1}\right) \quad \{\text{from eq}^n (4)\}$$

Q.2 Show that if p is a positive Integer then there exist $a, b > 0$ such that $|E(z,p)| \leq b \exp(a|z|^p)$

Sol By Weierstrass primary factor we have,

$$E(z,p) = (1-z) \exp\left\{\sum_{n=1}^{p} \frac{z^n}{n}\right\}$$

$$\text{so, } |E(z,p)| \leq (1+|z|) \exp\left\{\sum_{n=1}^{p} \frac{|z|^n}{n}\right\} \quad \text{--- (1)}$$

Now,

$$\exp|z| = 1 + |z| + \frac{|z|^2}{\lfloor 2} + \cdots$$

$$\geq 1 + |z| \quad \text{--- (2), for large } |z|$$

Also for large $|z|$,

$$\frac{|z|^p}{p} \geq \frac{|z|^{p-1}}{p-1} \geq \frac{|z|^{p-2}}{p-2} \geq \cdots \geq |z|$$

$$\text{Then, } p \cdot \left(\frac{|z|^p}{p}\right) \geq \sum_{n=1}^{p} \frac{|z|^n}{n}$$

$$\Rightarrow |z|^p \geq \sum_{n=1}^{p} \frac{|z|^n}{n} \quad \text{--- (3)}$$

From (1), (2) and (3), we have

$$|E(z,p)| \leq (\exp|z|) \exp(|z|^p)$$

$$= \exp(|z| + |z|^p)$$

$$\leq \exp(2|z|^p), \text{ for large } |z| \text{ say } |z| > R$$

If $|z| \leq R$ then we choose $b > 1$ such that

$$|E(z,p)| \leq b \exp(2|z|^p)$$

Choosing $a \geq 2$

$$\boxed{|E(z,p)| \leq b \exp(a|z|^p)}$$

24/01/20

Gamma Function

We construct a Gamma function which is meromorphic with poles at non-positive integers, i.e., $z = 0, -1, -2, \ldots$

For this assignment, we introduce functions which have only negative zeros. The simplest function of this kind is

$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \quad \text{--- (1)}$$

Then $$G(-z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \quad \text{--- (2)}$$

Obviously the function $G(-z)$ has only positive zeros

Now,

$$z \cdot G(z) G(-z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \frac{\sin \pi z}{\pi} \quad \text{--- (3)}$$

Now, $$G(z-1) = \prod_{n=1}^{\infty} \left(1 + \frac{z-1}{n}\right) e^{-\frac{z-1}{n}} \quad \text{--- (4)}$$

Zeros of $G(z-1)$ are $0, -1, -2, -3, \ldots$

Thus, $G(z-1)$ has the same zeros as $G(z)$ and in addition a simple zero at the origin.

Therefore, by Weierstrass factorisation theorem, we can write

$$G(z-1) = z e^{\gamma(z)} G(z) \quad \text{— (A)}$$

where $\gamma(z)$ is an entire function

$$\prod_{n=1}^{\infty} \left(\frac{z-1+n}{n}\right) e^{-(z-1)/n} = z e^{\gamma(z)} \prod_{n=1}^{\infty} \left(\frac{z+n}{n}\right) e^{-z/n}$$

Taking logarithm both sides,

$$\sum_{n=1}^{\infty} \left[\log(z-1+n) - \log n - \left(\frac{z-1}{n}\right)\right] = \log z + \gamma(z) + \sum_{n=1}^{\infty} \left[\log(z+n) - \log n - \frac{z}{n}\right]$$

Differentiating wrt z, we get

$$\sum_{n=1}^{\infty} \left[\frac{1}{z-1+n} - \frac{1}{n}\right] = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left[\frac{1}{z+n} - \frac{1}{n}\right] \quad \text{— (5)}$$

Replacing n by $n+1$, then the LHS of (5) becomes

$$\sum_{n=1}^{\infty} \left[\frac{1}{z-1+n} - \frac{1}{n}\right] = \sum_{n=0}^{\infty} \left[\frac{1}{z+n} - \frac{1}{n+1}\right]$$

$$= \left(\frac{1}{z} - 1\right) + \sum_{n=1}^{\infty} \left[\frac{1}{z+n} - \frac{1}{n+1}\right]$$

$$= \left(\frac{1}{z} - 1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$\left\{\because \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1\right\}$$

$$= \left(\frac{1}{z} - 1\right) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) + 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \left[\frac{1}{z-1+n} - \frac{1}{n}\right] = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right) \quad \text{--- } (6)$$

On comparing eqⁿ (5) and (6), we get

$$\gamma'(z) = 0$$

$\therefore$ $\gamma(z)$ is a constant, so we denote $\gamma(z)$ by γ

By eqⁿ (A), we obtain

$$G(z-1) = z e^{\gamma} G(z) \quad \text{--- } (7)$$

Put $z = 1$ in eqⁿ (7), we get

$$G(0) = e^{\gamma} G(1)$$

From eqⁿ (1), $G(0) = 1$

and $G(1) = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}$

$$\therefore \quad 1 = e^{\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}$$

$$\Rightarrow e^{-\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n} \quad \text{--- } (8)$$

Here the n^{th} partial product is,

$$\left(1 + \frac{1}{1}\right) e^{-1} \cdot \left(1 + \frac{1}{2}\right) e^{-1/2} \cdots\cdots \left(1 + \frac{1}{n}\right) e^{-1/n}$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots\cdots \frac{n+1}{n} \cdot e^{-1} \cdot e^{-1/2} \cdots\cdots e^{-1/n}$$

$$= (n+1) e^{-\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)}$$

On taking logarithm, (taking $\lim_{n\to\infty}$ in eq^n(8) and then taking log)

$$-\gamma = \lim_{n\to\infty}\left[\log(n+1) - \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]$$

$$-\gamma = \lim_{n\to\infty}\left[\log n - \left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}\right)\right]$$

$$\gamma = \lim_{n\to}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \log n\right]$$

The constant γ is called Euler's constant and its approximate value is 0.57722.

We now consider the function $H(z) = e^{\gamma z} G(z)$ —(9)

Now, $H(z-1) = e^{\gamma(z-1)} G(z-1)$

$$= e^{\gamma(z-1)} \cdot z e^{\gamma} G(z) \qquad (\text{from eq}^n\ 7)$$

$$= z \cdot e^{\gamma z} G(z)$$

$$\boxed{H(z-1) = z \cdot H(z)} \qquad (\text{from } 9)$$

Euler's Gamma function

Euler's Gamma function $\Gamma(z)$ is defined as

$$\Gamma(z) = \frac{1}{zH(z)}$$

$$= \frac{1}{z \cdot e^{\gamma z} G(z)} \qquad (\text{from } 9)$$

$$= \frac{e^{-\gamma z}}{z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right) e^{-z/n}} \qquad (\text{from } (1))$$

$$= \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$

Note:- We observed that $\Gamma(z)$ is well-defined in the whole complex plane except for $z=0,-1,-2,-3,\ldots$ which are simple poles of the function. Hence $\Gamma(z)$ is meromorphic with this poles but has no zeros.

Properties of Gamma function

(I) $\Gamma(z+1) = z\Gamma(z)$

Proof:- By the def^n of Euler's Gamma function, we have

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \quad \text{--- (1)}$$

Consider the function,

$$G(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n} \quad \text{--- (2)}$$

and $H(z) = e^{\gamma z} G(z)$

Then, $z H(z) = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$

$$\Rightarrow \frac{1}{zH(z)} = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$

$$\Rightarrow \frac{1}{zH(z)} = \Gamma(z) \quad \text{--- (3)}$$

$$\therefore \Gamma(z+1) = \frac{1}{(z+1)H(z+1)} \quad \text{--- (4)}$$

Now, since we have the relation

$$H(z-1) = z\,H(z) \quad \text{--- (5)}$$

$$\therefore \quad H(z) = (z+1)\,H(z+1) \quad \text{--- (6)}$$

Thus, from eq^n (3) & (5),

$$\Gamma(z) = \frac{1}{H(z-1)} \quad \text{--- (7)}$$

Also from (4) & (6)

$$\Gamma(z+1) = \frac{1}{H(z)} = \frac{1}{\left(\frac{H(z-1)}{z}\right)} = \frac{z}{H(z-1)}$$

$$\Rightarrow \boxed{\Gamma(z+1) = z.\Gamma z}$$

(II) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof:- By def^n of Euler's Gamma function, we have

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \quad \text{--- (1)}$$

and consider the function,

$$G(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n} \quad \text{--- (2)}$$

Combining eq^n (1) and (2)

$$\Gamma(z) = \frac{e^{-\gamma z}}{z\,G(z)} \quad \text{--- (3)}$$

Also we have the property, $\Gamma(z+1) = z\,\Gamma(z)$

Writing $z = -z$, we get

$$\Gamma(1-z) = -z\,\Gamma(-z)$$

$$= -z \frac{e^{\gamma z}}{(-z)\,G(-z)}$$

$$\Rightarrow \Gamma(1-z) = \frac{e^{\gamma z}}{G(-z)} \quad \text{—— (4)}$$

Multiplying eq^n (3) and (4), we get

$$\Gamma(z)\,\Gamma(1-z) = \frac{1}{z\,G(z)\,G(-z)}$$

$$= \frac{\pi}{\sin z\pi}$$

Put $z = 1/2$

$$\Gamma\tfrac{1}{2}\,\Gamma\tfrac{1}{2} = \frac{\pi}{\sin \pi/2}$$

$$\Rightarrow \boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

(III) $\Gamma(n+1) = \lfloor n$

(IV) <u>Legendre's</u> <u>Duplication</u> <u>Formula</u>

$$\sqrt{\pi}\,\Gamma(2z) = 2^{(2z-1)}\,\Gamma z\,\Gamma(z+\tfrac{1}{2})$$

<u>Proof</u>:- By the def^n of Euler's Gamma function, we have

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \quad \text{—— (1)}$$

Taking log,

$$\log \Gamma(z) = \left(\log e^{-\gamma z} - \log z\right) + \sum_{n=1}^{\infty} \left[\log\left(1+\frac{z}{n}\right)^{-1} + \log e^{z/n}\right]$$

$$\log \Gamma(z) = -\gamma z - \log z + \sum_{n=1}^{\infty} \left[-\log\left(1+\frac{z}{n}\right) + \frac{z}{n}\right]$$

Differentiating wrt z, we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left[-\frac{1}{(1+z/n)} \cdot \frac{1}{n} + \frac{1}{n}\right]$$

$$\Rightarrow \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{-1}{(z+n)} + \frac{1}{n}\right]$$

Differentiating again wrt z,

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \left[\frac{1}{(n+z)^2} + 0\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} \quad \text{---} (2)$$

Similarly, we can obtain

$$\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right) = 2 \sum_{n=0}^{\infty} \frac{1}{(2z+n)^2} \quad \text{---} (3)$$

and $$\frac{d}{dz}\left(\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right) = \sum_{n=0}^{\infty} \frac{1}{\left(z+\frac{1}{2}+n\right)^2} \quad \text{---} (4)$$

Adding eqn (2) and (4),

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) + \frac{d}{dz}\left(\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2} + \sum_{n=0}^{\infty} \frac{1}{(z+1/2+n)^2}$$

$$= 4\sum_{n=0}^{\infty} \frac{1}{(2n+2z)^2} + 4\sum_{n=0}^{\infty} \frac{1}{(2z+2n+1)^2}$$

$$= 4\sum_{n=0}^{\infty} \frac{1}{(2z+n)^2}$$

$$= 2\frac{d}{dz}\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)$$

Integrating wrt z, we get

$$\frac{\Gamma'(z)}{\Gamma(z)} + \frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)} = \frac{2\Gamma'(2z)}{\Gamma(2z)} + a$$

Again integrating wrt z,

$$\log\Gamma(z) + \log\Gamma(z+1/2) = \log\Gamma(2z) + az + b$$

$$\Rightarrow \log\left(\Gamma(z).\Gamma(z+1/2)\right) = \log\Gamma(2z)\, e^{az+b}$$

$$\Rightarrow \Gamma(z).\Gamma(z+1/2) = \Gamma(2z).\, e^{az+b} \quad \text{——} (5)$$

To determine a and b, putting $z=1/2$ and $z=1$ in eq^n (5),

$$\Gamma 1/2\ \Gamma 1 = \Gamma 1\ e^{a/2+b}$$

$$\Rightarrow \sqrt{\pi} = e^{a/2+b} \quad \text{——} (6)$$

And,
$$\Gamma 1 \, \Gamma 3/2 = \Gamma 2 \, e^{a+b}$$
$$\frac{1}{2}\sqrt{\pi} = e^{a+b}$$

Taking log in eqⁿ (6) and (7),

$$\log \pi = a+2b \quad \text{and} \quad \frac{1}{2}\log\pi - \log 2 = a+b$$

$$\Rightarrow b = \frac{1}{2}\log\pi + \log 2 \quad \text{and} \quad a = 2\log 2$$

Therefore from eqⁿ (5),

$$\Gamma(z)\,\Gamma z+1/2 = \Gamma(2z)\, e^{-(2\log 2)z + \frac{1}{2}\log\pi + \log 2}$$

$$= \Gamma(2z)\, e^{\log 2(-2z+1) + \frac{1}{2}\log\pi}$$
$$= \Gamma(2z)\, e^{\log 2(-2z+1)} \cdot e^{\log\sqrt{\pi}}$$

$$= \Gamma(2z)\, e^{-\log 2^{2z}} \cdot e^{\log 2} \cdot \sqrt{\pi}$$

$$= \Gamma(2z)\, 2^{-(2z+1)}\sqrt{\pi}$$

$$\boxed{2^{(2z-1)}\,\Gamma z\,\Gamma(z+1/2) = \sqrt{\pi}\,\Gamma(2z)}$$

(V) $$\Gamma(z) = \lim_{n\to\infty} \frac{\lfloor n \cdot n^z}{z(z+1)\ldots(z+n)}$$

Proof: By defⁿ of Euler's Gamma function, we have

$$\Gamma z = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1} e^{z/k} \quad \text{--- (1)}$$

Then, $(\Gamma(z))^{-1} = z \exp(\gamma z) \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right) \exp\left(\frac{-z}{k}\right) \right\}$ —(2)

we have,

$$\gamma = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{1}{k} - \log n \right\}$$

Substituting γ in eq^n (2),

$$(\Gamma(z))^{-1} = z \exp\left\{ \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) z \right\} \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right) \exp\left(\frac{-z}{k}\right) \right\}$$

$$= \lim_{n\to\infty} \left[z \exp\left\{ z \sum_{k=1}^{n} \frac{1}{k} - z \log n \right\} \right] \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right) \exp\left(\frac{-z}{k}\right) \right\}$$

$$= \lim_{n\to\infty} \left[z \exp\left(z \sum_{k=1}^{n} \frac{1}{k} \right) . \exp\left(-z \log n\right) \right] \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right) \exp\left(\frac{-z}{k}\right) \right\}$$

$$= \lim_{n\to\infty} \left[z \exp\left(z \sum_{k=1}^{n} 1/k \right) \exp\left(-z \log n\right) \right] \left\{ \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) \right\} \left\{ \prod_{k=1}^{n} \exp\left(\frac{-z}{k}\right) \right\}$$

$$= \lim_{n\to\infty} \left[z \exp\left(z \sum_{k=1}^{n} \frac{1}{k} \right) \exp\left(-z \log n\right) \right] \left\{ \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) \right\} \left\{ \exp\left(-z \sum_{k=1}^{n} 1/k\right) \right\}$$

$$= \lim_{n\to\infty} \left[z \exp\left(-z \log n\right) \right] \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)$$

$$= \lim_{n\to\infty} \left[z\, e^{\log n^{-z}} \right] \prod_{k=1}^{n} \left(1+\frac{z}{k}\right)$$

$$= \lim_{n\to\infty} \left[z\, n^{-z} \prod_{k=1}^{n} \left(1+\frac{z}{k}\right) \right]$$

$$= \lim_{n\to\infty} \left[z\, n^{-z} \left(1+\frac{z}{1}\right) \left(1+\frac{z}{2}\right) \left(1+\frac{z}{3}\right) \cdots \left(1+\frac{z}{n}\right) \right]$$

$$(\Gamma(z))^{-1} = \lim_{n\to\infty} \left[z\, n^{-z} \left(\frac{1+z}{1}\right) \left(\frac{2+z}{2}\right) \left(\frac{3+z}{3}\right) \cdots \left(\frac{n+z}{n}\right) \right]$$

$$\Gamma(z) = \lim_{n\to\infty} \left[\frac{n^{z}}{z} \frac{\lfloor n}{(z+1)(z+2)\cdots(z+n)} \right]$$

$$\boxed{\Gamma(z) = \lim_{n\to\infty} \frac{\lfloor n \; n^{z}}{z(z+1)\cdots(z+n)}}$$

Date
Page

(VI) Residue at the poles of $\Gamma(z)$

We have, $\Gamma(z) = \dfrac{\Gamma(z+1)}{z}$

By the repeated application of formula,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \frac{\Gamma(z+n+1)}{z(z+1)\ldots(z+n)}$$

$$\Rightarrow \Gamma z = \frac{\Gamma(z+n+1)}{z(z+1)\ldots(z+n)}$$

Now Γz has simple poles at $z = 0, -1, -2, -3, \ldots, -n, \ldots$
Hence the residue at $z = -n$

$$\operatorname*{Res}_{z=-n} \Gamma(z) = \lim_{z \to -n} \frac{(z+n)\Gamma(z+n+1)}{z(z+1)\ldots(z+n)}$$

$$= \lim_{z \to -n} \frac{\Gamma(z+n+1)}{z(z+1)\ldots(z+n-1)}$$

$$= \frac{\Gamma 1}{(-n)(-n+1)\ldots(-1)}$$

$$= \frac{1}{-(n)(n-1)\ldots 1}$$

$$= (-1)^n \frac{1}{\lfloor n}$$

$$\boxed{\operatorname*{Res}_{z=-n} \Gamma(z) = \frac{(-1)^n}{\lfloor n}}$$

H.W. **(VII) Analytic continuation of Gamma function**

Theorem:- $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt \qquad (\text{Re } z > 0)$

$s = z = \sigma + it$

30/01/20

Riemann Zetā function

The Riemann Zeta function $\zeta(s)$ is defined for Re $s > 1$ by the series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

which therefore represents an analytic function of s in the half plane.

Euler's theorem

Theorem:- For $\sigma = \text{Re } s > 1$, $\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} (1 - P_n^{-s})$

where $\{P_n\}$ is the sequence of prime numbers.

Proof:- Since the infinite product $\prod_{n=1}^{\infty} (1 - P_n^{-s})$ converges uniformly for $\sigma \geq \sigma_0 > 1$, if the same is true for the series $\sum_{n=1}^{\infty} |P_n^{-s}| = \sum_{n=1}^{\infty} P_n^{-\sigma}$

The series $\sum_{n=1}^{\infty} P_n^{-\sigma}$ is obtained by omitting the terms from the series $\sum_{n=1}^{\infty} n^{-\sigma}$

By the assumption $\sigma > 1$, we have

$$\zeta(s)(1 - 2^{-s}) = \left(\sum_{n=1}^{\infty} n^{-s}\right)(1 - 2^{-s})$$

$$= \sum_{n=1}^{\infty} n^{-s} - \sum_{n=1}^{\infty} (2n)^{-s}$$

$$= \sum m^{-s}$$, where m runs through the odd integers

By the same reasoning,

$$\zeta(s)(1-2^{-s})(1-3^{-s}) = \Sigma m^{-s}$$

where this time m runs through all integers that are neither divisible by 2, nor 3

Continuing in this way, we get

$$\zeta(s)(1-2^{-s})(1-3^{-s})\ldots(1-P_N^{-s}) = \Sigma m^{-s}$$

The sum of the right being over all integers that contain none of the prime factors $2, 3, \ldots, P_N$

The first term in the sum is 1 and the next is P_{N+1}^{-s}

The sum of all terms except the first tends to 0 as $N \to \infty$

We obtain,

$$\lim_{N \to \infty} \zeta(s) \prod_{n=1}^{N} (1-P_N^{-s}) = 1$$

$$\Rightarrow \zeta(s) \prod_{n=1}^{\infty} (1-P_N^{-s}) = 1$$

$$\Rightarrow \frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} (1-P_N^{-s})$$

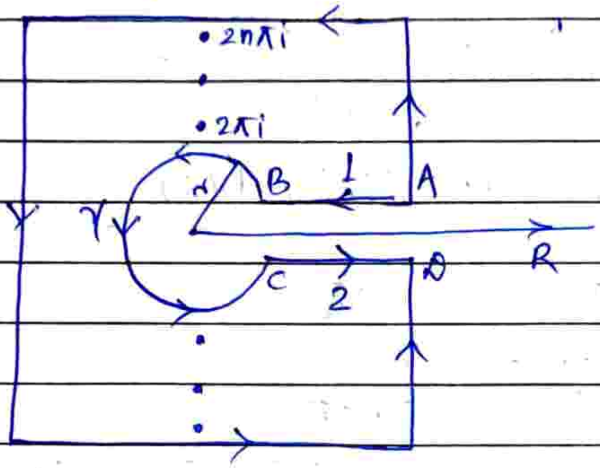
Proved

31/01/20

Theorem:- For $\sigma > 1$, $\zeta(s) = -\dfrac{\Gamma(1-s)}{2\pi i} \displaystyle\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz$

where $(-z)^{s-1}$ is defined on the complement of the positive real axis as $e^{(s-1)\log(-z)}$ with $-\pi < \operatorname{Im} \log(-z) < \pi$

Proof:-



The integral $\displaystyle\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz$ is convergent.

By Cauchy's theorem its value does not depend on the shape of C as long as C does not enclose any multiples of $2\pi i$

When $r \to 0$, line 1 and line 2 coincides with $\mathbb{R}$

On upper edge $(-z)^{s-1} = x^{s-1} e^{-(s-1)\pi i}$

On lower edge $(-z)^{s-1} = x^{s-1} e^{(s-1)\pi i}$

We obtain,

$$\int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz = \int_{AB} \frac{(-z)^{s-1}}{e^z - 1}\, dz + \int_\gamma \frac{(-z)^{s-1}}{e^z - 1}\, dz + \int_{CD} \frac{(-z)^{s-1}}{e^z - 1}\, dz$$

$$\Rightarrow \int_C \frac{(-z)^{s-1}}{e^z - 1}\, dz = -\int_0^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1}\, dx + \int_0^\infty \frac{x^{s-1} e^{(s-1)\pi i}}{e^x - 1}\, dx$$

$s = \sigma + it$

$$= \int_0^\infty \frac{x^{s-1}}{e^x - 1} \left\{ e^{(s-1)\pi i} - e^{-(s-1)\pi i} \right\} dx$$

$$= \int_0^\infty \frac{x^{s-1}}{e^x - 1} \left(2i \sin (s-1)\pi \right) dx$$

$$= 2i \sin (s-1)\pi \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

$$= 2i \sin (s-1)\pi \; \zeta(s) \cdot \Gamma(s)$$

But $\sin (s-1)\pi = -\sin s\pi$
and $\Gamma(s)\,\Gamma(1-s) = \dfrac{\pi}{\sin s\pi}$

$$\therefore \int_C \frac{(-z)^{s-1}}{e^z - 1} dz = -2i \sin s\pi \; \frac{\zeta(s) \cdot \pi}{\Gamma(1-s) \cdot \sin s\pi}$$

$$\Rightarrow \quad \zeta(s) = \frac{-\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz$$

Proved

01/02/20

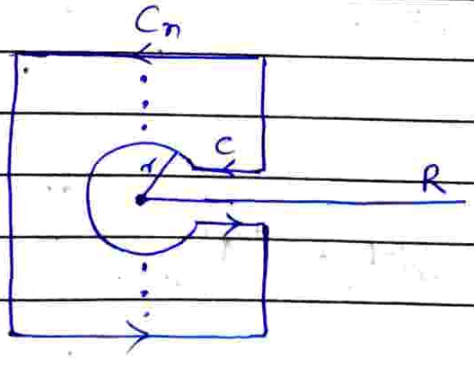
The Riemann Fundamental Equation

Statement :- For $\sigma < 0$, the following relation

$$\zeta(s) = 2^s \pi^{(s-1)} \sin \frac{\pi s}{2} \, \Gamma(1-s) \, \zeta(1-s)$$

is called the Riemann fundamental equation.

Proof :-



We consider the path C_n and assume that the square part lies on the lines $t = \pm(2n+1)\pi$ and $\sigma = \pm(2n+1)\pi$

The cycle $C_n - C$ has winding number 1 about the points $\pm 2m\pi i$, $m = 1, 2, \ldots, n$

At these points the function $\frac{(-z)^{s-1}}{e^z - 1}$ has simple poles with residues $(\mp 2m\pi i)^{s-1}$

Thus, By Cauchy's residue theorem

$$\frac{1}{2\pi i}\int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1}\, dz = \sum_{m=1}^{n}\left[-(2m\pi i)^{s-1} + (2m\pi i)^{s-1}\right]$$

$$= 2\sum_{m=1}^{n} (2m\pi)^{s-1} \sin\frac{\pi s}{2} \quad \text{---(1)}$$

We divide C_n into $C_n' + C_n''$, where C_n' is the part on the square and C_n'', the part outside the square.

Then we see that $|e^z - 1|$ is bounded below on C_n' by a fixed positive constant independent of n while $|(-z)^{s-1}|$ is bounded by multiple of $n^{\sigma - 1}$

The length of C_n' is of the order of n and we obtain,

$$\left|\int_{C_n'} \frac{(-z)^{s-1}}{e^z - 1}\, dz\right| \leq A n^{\sigma}, \text{ for some constant } A.$$

If $\sigma < 0$, then $\int_{C_n'} \frac{(-z)^{s-1}}{e^z - 1}\, dz \to 0$ as $n \to \infty$

~~And~~ and same for C_n''

$\therefore$ The integral $\displaystyle\int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1} dz \to \int_{-C} \frac{(-z)^{s-1}}{e^z-1} dz$

i.e. the integral over $C_n - C$ tends to the integral over $-C$.

But we have,

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z-1} dz$$

$$\therefore \quad \frac{1}{2\pi i} \int_{C_n - C} \frac{(-z)^{s-1}}{e^z-1} dz \to \frac{\zeta(s)}{\Gamma(1-s)}$$

Then from eq^n(1), we obtain

$$\frac{\zeta(s)}{\Gamma(1-s)} = 2 \sum_{m=1}^{n} (2m\pi)^{s-1} \sin \frac{\pi s}{2}$$

$$= 2 \sum_{m=1}^{n} 2^{s-1} m^{s-1} \pi^{s-1} \sin \frac{\pi s}{2}$$

$$= 2^s \pi^{s-1} \left(\sum_{m=1}^{n} m^{s-1} \right) \sin \frac{\pi s}{2} \quad \text{——(2)}$$

Now on the same condition $\sigma < 0$, the series $\sum_{m=1}^{n} m^{s-1}$ converges to $\zeta(1-s)$ as $n \to \infty$

From eq^n(2), we have

$$\frac{\zeta(s)}{\Gamma(1-s)} = 2^s \pi^{s-1} \zeta(1-s) \sin \frac{\pi s}{2}$$

$$\Rightarrow \boxed{\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s)\, \zeta(1-s) \sin \frac{\pi s}{2}}$$

This relation is known as the Riemann functional equation.

Runges Theorem

Lemma:- Let γ be a rectifiable curve and K be a compact set such that $K \cap \{\gamma\} = \phi$. Let f be a continuous function on $\{\gamma\}$ and let $\epsilon > 0$ be given. Then there is a rational function $R(z)$ having all its poles on $\{\gamma\}$ and such that

$$\left| \int_\gamma \frac{f(w)}{w-z} \, dw - R(z) \right| < \epsilon \quad \forall \; z \in K$$

Proof:- Since $K \cap \{\gamma\} = \phi$, so that $d(K, \{\gamma\}) > 0$
i.e. $\exists$ a number $0 < r < d(K, \{\gamma\})$

Let us define γ on $[0,1]$. Then $0 \leq s, t \leq 1$ and $z \in K$, we have

$$\left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(s))}{\gamma(s) - z} \right| = \left| \frac{f(\gamma(t))\gamma(s) - z f(\gamma(t)) - f(\gamma(s))\gamma(t) + z f(\gamma(s))}{(\gamma(t)-z)(\gamma(s)-z)} \right|$$

$$= \frac{\left| f(\gamma(t))\gamma(s) - f(\gamma(s))\gamma(t) - z\{f(\gamma(t)) - f(\gamma(s))\} \right|}{|\gamma(t) - z| \, |\gamma(s) - z|}$$

$$= \frac{\left| [f(\gamma(t))\gamma(s) - f(\gamma(t))\gamma(t)] - [f(\gamma(s))\gamma(t) - f(\gamma(t))\gamma(t)] - z[f(\gamma(t)) - f(\gamma(s))] \right|}{|\gamma(t) - z| \, |\gamma(s) - z|}$$

$$\leq \frac{1}{r^2} \Big[\left| f(\gamma(t))\gamma(s) - f(\gamma(t))\gamma(t) \right| + \left| f(\gamma(s))\gamma(t) - f(\gamma(t))\gamma(t) \right| + |z| \left| f(\gamma(t)) - f(\gamma(s)) \right| \Big]$$

$$= \frac{1}{r^2} |f(\gamma(t))| \, |\gamma(s)-\gamma(t)| + \frac{1}{r^2} |\gamma(t)| \, |f(\gamma(s))-f(\gamma(t))|$$

$$+ \frac{1}{r^2} |z| \, |f(\gamma(s)) - f(\gamma(t))|$$

$\longrightarrow$ (1)

Since K is compact, γ is a rectifiable curve and f is a continuous function on $\{\gamma\}$, therefore $\exists$ a constant $c > 0$ such that

(i) $|z| \leq c$, $\forall$ $z \in K$

(ii) $|\gamma(t)| \leq c$, $\forall$ $t \in [0,1]$

(iii) $|f(\gamma(t))| \leq c$, $\forall$ $t \in [0,1]$

Thus for all s and t in $[0,1]$ and z in K, we have

$$\left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z} \right| \leq \frac{c}{r^2} |\gamma(s)-\gamma(t)| + \frac{2c}{r^2} |f(\gamma(s)) - f(\gamma(t))|$$

$\longrightarrow$ (2)

Again, since both γ and $f \circ \gamma$ are uniformly continuous on $[0,1]$, therefore $\exists$ a partition $P = \{t_0, t_1, t_2, \ldots, t_n\}$ of $[0,1]$ such that,

$$0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = 1$$

and $$\left| \frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_j))}{\gamma(t_j)-z} \right| \leq \frac{\epsilon}{V(\gamma)} \quad \longrightarrow (3)$$

for $t_{j-1} \leq t \leq t_j$, $1 \leq j \leq n$, $z \in K$

$V(\gamma)$ denotes the total variation of γ defined by

$$V(\gamma) = \sup_{1\le j\le n}\left\{ \left| \gamma(t_j) - \gamma(t_{j-1})\right| \right\} \quad \text{--- (4)}$$

We now define $R(z)$ to be the rational function

$$R(z) = \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))\,[\gamma(t_j) - \gamma(t_{j-1})]}{[\gamma(t_{j-1}) - z]}$$

Clearly the poles of $R(z)$ are $\gamma(0), \gamma(t_1), \dots, \gamma(t_{n-1})$.

Now, using (4), we compute

$$\left| \int_\gamma \frac{f(w)}{w-z}\,dw - R(z) \right| = \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{f(\gamma(t))}{\gamma(t)-z}\,d\gamma(t) - \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))\,[\gamma(t_j) - \gamma(t_{j-1})]}{[\gamma(t_{j-1}) - z]} \right|$$

$$= \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1})-z} \right] d\gamma(t) \right|$$

$$< \frac{\epsilon}{V(\gamma)} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} d|\gamma|(t) = \frac{\epsilon}{V(\gamma)} \cdot V(\gamma)$$

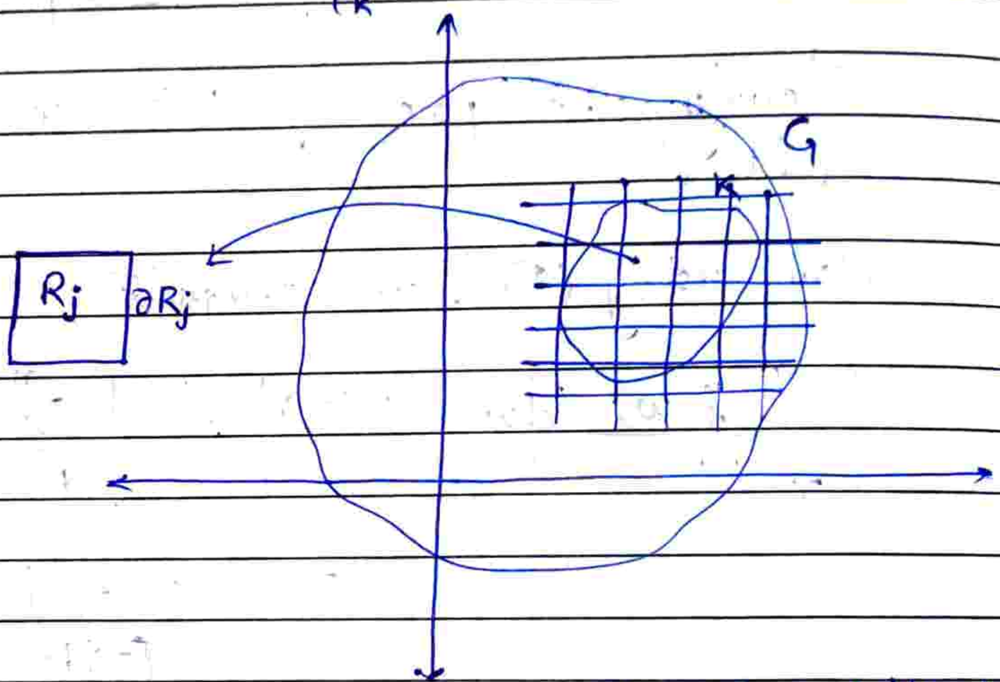
$$< \epsilon \qquad \forall\ z \in K.$$

Hence proved

Imp
Proposition:- Let K be a compact subset of the region G, then there are straight line segments $\gamma_1, \gamma_2, \dots, \gamma_n$ in $G-K$ such that for every function f in $H(G)$

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(\omega)}{\omega - z} \, d\omega \quad , \forall z \in K$$

Proof:-



Suppose that the region K is enlarged a little by assuming that $K = \overline{\text{int } K}$

Let $0 < \delta < \frac{1}{2} d(K, \mathbb{C}-G)$

Now we construct a grid of horizontal and vertical lines in the plane such that consecutive lines are less than a distance δ.
Let the rectangles intersecting K be $R_1, R_2, \dots, R_m$
Also, let ∂R_j be the boundary of the rectangle R_j, $1 \leq j \leq m$.

Consider a polygon with the counter clockwise direction.

Suppose $z \in R_j$, $1 \le j \le m$, then
$d(z, K) < \sqrt{2}\,\delta$ so that $R_j \subset G$

Suppose R_i and R_j have a common side and let σ_i and σ_j be the line segments in ∂R_i and ∂R_j such that

$$R_i \cap R_j = \{\sigma_i\} = \{\sigma_j\}$$

According to the direction given. ∂R_i and ∂R_j, σ_i and σ_j are directed in opposite sense

so,
if φ is any continuous function on $\{\sigma_j\}$ then we have

$$\int_{\sigma_j} \varphi = \int_{-\sigma_i} \varphi \implies \int_{\sigma_j} \varphi + \int_{\sigma_i} \varphi = 0$$

Suppose $\gamma_1, \gamma_2, \ldots, \gamma_n$ are those directed line segments that constitute a side of exactly one of the R_j, $1 \le j \le m$ then

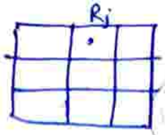
$$\sum_{k=1}^{n} \int_{\gamma_k} \varphi = \sum_{j=1}^{m} \int_{\partial R_j} \varphi \quad \text{———} (1)$$

holds for every continuous function φ on $\bigcup_{j=1}^{m} \partial R_j$

We now claim that each γ_k is in $G - K$.

If one of the γ_k intersects K then there are two rectangles in the grid with γ_k as a side and so both meet K, i.e., γ_k is the common side of two of the rectangles $R_1, R_2, \ldots, R_m$ and this contradicts the choice of γ_k.

inside for R_j
outside for every other R



Further, if $z \in K$ and is not on the boundary of any R_j, then

$$\varphi(w) = \frac{1}{2\pi i}\left[\frac{f(w)}{w-z}\right]$$

is a continuous function on $\bigcup_{j=1}^{m} \partial R_j$ for $f \in H(G)$

Thus it follows from eq^n(1), that,

$$\sum_{j=1}^{m} \frac{1}{2\pi i}\int_{\partial R_j} \frac{f(w)}{w-z}\,dw = \sum_{k=1}^{m} \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(w)}{w-z}\,dw \quad \text{——(2)}$$

But z belongs to the interior of exactly one R_j

If $z \notin R_j$, we have

$$\frac{1}{2\pi i}\int_{\partial R_j} \frac{f(w)}{w-z}\,dw = 0$$

and if $z \in R_j$, then by Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_{\partial R_j} \frac{f(w)}{w-z}\,dw = f(z)$$

Thus from (2),

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(w)}{w-z}\,dw \quad \text{——(3)}$$

where $z \in K - \bigcap_{j=1}^{m} \partial R_j$

But we see that both sides of (3) are continuous functions on K because each γ_k misses K and they agree on the dense

subset of K.
Hence (3) holds for all $z \in K$.

Hence proved.

Runge's Theorem

Statement :- Let K be a compact subset of $\mathbb{C}$ and let E be a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$. If f is analytic in an open set G containing K and $\epsilon > 0$ then there is a rational function $R(z)$ whose only poles lie in R and such that $|f(z) - R(z)| < \epsilon$, $\forall z \in K$

Proof :- Suppose f is analytic in an open set G containing K.

Now, we list some results which are required to prove the theorem,

Result 1 :- Proposition with proof.

Result 2 :- Lemma

Result 3 :- If $a \in \mathbb{C} - K$, then $(z-a)^{-1} \in B(E)$, where $B(E)$ = set of all functions in $C(K, \mathbb{C})$ such that there is a sequence $\{R_n\}$ of rational functions with poles in E such that $\{R_n\}$ converges uniformly to f on K.

From result 1 and result 2 we conclude that, there exist a rational function $R(z)$ with poles in $C-K$ such that

$$|f(z) - R(z)| < \epsilon \quad \forall z \in K$$

And result (3) shows that, since $B(E)$ is an algebra so that $R \in B(E)$

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Miltag Leffler's theorem

Let $\{b_n\}$ be sequence of complex numbers with $\lim_{n\to\infty} b_n = \infty$ and let $P_n(\zeta) = a_{n_1}\zeta + a_{n_2}\zeta^2 + \cdots + \cancel{a_{n_k}}\ a_{n_{k_n}}\zeta^{k_n}$, $n = 1, 2, \ldots$ be arbitrary polynomials of degree 1 and having no constant terms then there are functions which are meromorphic in whole plane with poles at the point b_n and the corresponding principal part

$$P_n\left(\frac{1}{z-b_n}\right) = \frac{a_{n_1}}{z-b_n} + \frac{a_{n_2}}{(z-b_n)^2} + \cdots + \frac{a_{n_{k_n}}}{(z-b_n)^{k_n}}$$

Moreover the most general meromorphic function of this kind can be written in the form

$$f(z) = \sum_{n=1}^{\infty}\left[P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right] + g(z)$$

where $p_n(z)$ are suitably choosen polynomials and $g(z)$ is an entire function.

Proof:- Let $P = \{b_1, b_2, \ldots\} \subset \mathbb{C}$

Without loss of generality, we may assume that $0 \notin P$ and P is infinite

Since $P_n\left(\frac{1}{z-b_n}\right)$ is analytic, $|z| < |b_n|$, therefore we can expand it in a Taylor series about the origin.

So, let $P_n\left(\frac{1}{z-b_n}\right) = \sum_{j=0}^{\infty} d_{nj} z^j$

Let $p_n(z)$ be the partial sum of this expansion upto degree r_n, so that

$$p_n(z) = \sum_{j=0}^{r_n} d_{nj} z^j$$

We choose r_n sufficiently large to suit our purpose, consider the remainder term

$$f_n(z) = P_n\left(\frac{1}{z-b_n}\right) - p_n(z)$$

We now estimate the remainder $P_n - p_n$

Let $M = M_n = \max\left\{ \left| P_n\left(\frac{1}{z-b_n}\right)\right| : |z| \le \frac{1}{2}|b_n| \right\}$

$\rho = \frac{1}{2}|b_n|$, $r = \frac{1}{4}|b_n|$

Then by use of explicit expression for theorem we have,

$$|f_n(z)| = \left| P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right| \le \frac{M\rho}{\rho - r}\left(\frac{r}{\rho}\right)^{r_n+1}$$

{By Taylor's theorem}

$$= \frac{M_n \frac{1}{2}|b_n|}{\frac{1}{2}|b_n| - \frac{1}{4}|b_n|}\left(\frac{\frac{1}{4}|b_n|}{\frac{1}{2}|b_n|}\right)^{r_n+1}$$

$$= \frac{M_n \frac{1}{2}|b_n|}{\frac{1}{4}|b_n|}\left(\frac{1}{2}\right)^{r_n+1}$$

$$= 2M_n\left(\frac{1}{2}\right)^{r_n+1}$$

$2M_n\left(\frac{1}{2}\right)^{r_n+1} \to 0$ as $r_n+1 \to \infty$

Now, since the series $\sum_{n=1}^{\infty} M_n$ is convergent

$\therefore$ by Weierstrass M-test, $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely

Let $h(z) = \sum_{n=1}^{\infty} f_n(z)$

i.e. $h(z) = \sum_{n=1}^{\infty}\left[P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right]$

And it converges absolutely and only singularity of $h(z)$ in complex plane are $z=b_n$, for $n=1,2,\ldots$

Finally suppose that there is a function f having the same poles and having principle part then the function

$$g(z) = f(z) - h(z)$$

having only removable singularity, so $g(z)$ is entire function

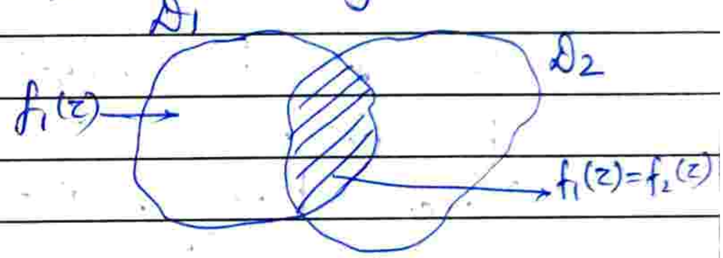
Thus we obtain $f(z) = h(z) + g(z)$

$$f(z) = \sum_{n=1}^{\infty}\left[P_n\left(\frac{1}{z-b_n}\right) - p_n(z)\right] + g(z)$$

Proved

** # Analytic Continuation

Let $f_1(z)$ be analytic in a domain D_1 and suppose we can find a function $f_2(z)$ which is analytic in domain D_2 such that D_1 and D_2 have an intersection, i.e. $D_1 \cap D_2 \neq \phi$
If $f_1(z) \equiv f_2(z)$ $\forall$ $z \in D_1 \cap D_2$, then we say that $f_2(z)$ is the analytic continuation of $f_1(z)$ from D_1 into D_2 via $D_1 \cap D_2$ or we may say that f_2 is analytic continuation of f_1



Note :- An analytic function f with its domain D is called a function element and is denoted by (f, D) if z is an element of D then (f, D) is called a function element of z

Direct Analytic Continuation $(f_1, D_1) \sim (f_2, D_2)$

Function elements (f_1, D_1), (f_2, D_2) are direct analytic continuation of each other iff

1) $D_1 \cap D_2 \neq 0$
2) $f_1(z) = f_2(z)$ $\forall$ $z \in D_1 \cap D_2$

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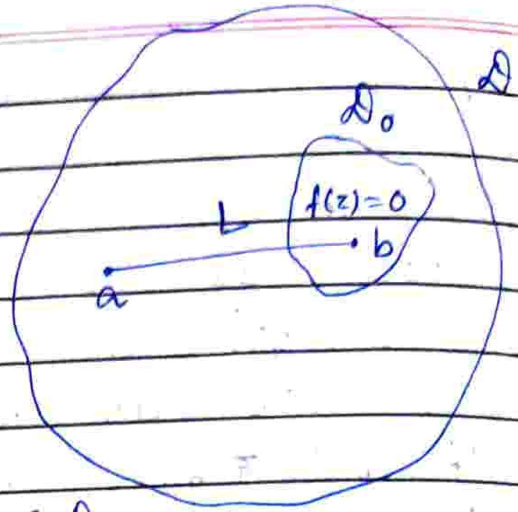
Theorem:- Let $f(z)$ be analytic in a domain D and let $f(z)$ vanish over a domain D_0 which is a part of D then $f(z)$ vanish over the whole domain D.

Proof:- We shall establish the theorem by contradiction.



Suppose if possible there exist $a \in \mathcal{D} - \mathcal{D}_0$ such that $f(a) \neq 0$

Then we take a point $b \in \mathcal{D}_0$ and join b with a by means of an arc L lying entirely in $\mathcal{D}$

Since $f(z)$ is analytic in $\mathcal{D}$, it is continuous at the point a.
Note that $f(a) \neq 0$
Therefore there are points near 'a' where $f(z)$ does not vanish.

Then $\exists$ a point c on the arc L such that near c. $f(z)$ vanishes at points on the arc on one side towards b and does not vanish on the other side towards a, which shows that $f(z)$ is not continuous at the point c which contradicts the hypothesis that $f(z)$ is analytic everywhere in $\mathcal{D}$.

And hence the theorem holds.

Uniqueness of Analytic Continuation

Theorem:- There cannot be more than one continuation of an analytic function $f(z)$ into the same domain.

Proof: Let $f_1(z)$ be analytic in the domain $\mathcal{D}_1$ and suppose that $f_2(z)$ is the analytic continuation of $f_1(z)$ into the domain $\mathcal{D}_2$ via $\mathcal{D}_1 \cap \mathcal{D}_2$

$$\therefore \quad f_1(z) = f_2(z) \quad \forall \; z \in \mathcal{D}_1 \cap \mathcal{D}_2 \quad \text{--- (1)}$$

Then we have to show that $f_2(z)$ is unique in $\mathcal{D}_2$

For, let $g_2(z)$ be another analytic continuation of $f_1(z)$ into the domain $\mathcal{D}_2$ via $\mathcal{D}_1 \cap \mathcal{D}_2$

i.e., $f_1(z) = g_2(z) \quad \forall \; z \in \mathcal{D}_1 \cap \mathcal{D}_2$ --- (2)

From (1) & (2)

$$f_2(z) = g_2(z) \quad \forall \; z \in \mathcal{D}_1 \cap \mathcal{D}_2 \quad \text{--- (3)}$$

$\Rightarrow \quad f_2(z) - g_2(z) = 0 \quad \forall \; z \in \mathcal{D}_1 \cap \mathcal{D}_2$

$\Rightarrow \quad (f_2 - g_2)(z) = 0 \quad \forall \; z \in \mathcal{D}_1 \cap \mathcal{D}_2$

$\Rightarrow \quad (f_2 - g_2)(z) = 0 \quad \forall \; z \in \mathcal{D}_2$ {by previous theorem}

$\Rightarrow \quad f_2(z) - g_2(z) = 0 \quad \forall \; z \in \mathcal{D}_2$

$\Rightarrow \quad f_2(z) = g_2(z) \quad \forall \; z \in \mathcal{D}_2$

Hence analytic continuation is unique.

** # Schwarz's Reflection Principle

Statement: Let $f(z)$ be a function of z analytic in a domain $\mathcal{D}$ which contains a segment of X-axis about which $\mathcal{D}$ is symmetric. Then $\overline{f(z)} = f(\bar{z})$, for all $z \in \mathcal{D}$.

Proof: We know, $\overline{f(z)} = f(z)$

Theorem:- In other words, $f(z)$ takes conjugate values for conjugate values of z iff $f(x)$ is real for each point on the segment of X-axis.

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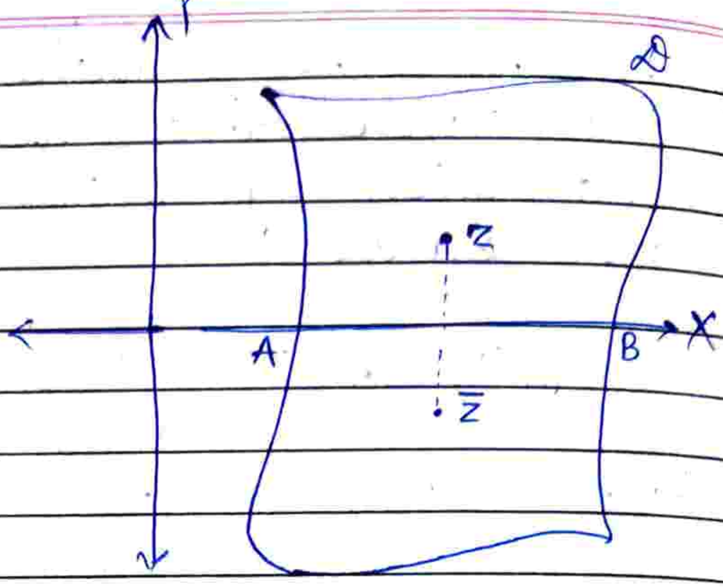
$\therefore$ the condition
$\overline{f(z)} = f(\bar{z})$
is equivalent to the condition
$\overline{f(\bar{z})} = f(z)$ —— (1)



Suppose the domain $\mathscr{D}$ have the segment AB of the real axis in ~~the~~ its interior and let $\mathscr{D}$ be symmetrical about the segment AB.
Assume that $f(x)$ is real for each x in AB.

Then we have to show that condition (1) holds.

We write $g(z) = \overline{f(\bar{z})}$ —— (2)

$$\left.\begin{aligned} \text{and } f(z) &= u(x,y) + iv(x,y) \\ g(z) &= p(x,y) + iq(x,y) \end{aligned}\right\} \text{—— (3)}$$

Then $f(\bar{z}) = u(x,-y) + iv(x,-y)$

$\therefore \overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$ —— (4)

From (2), (3) & (4)
$g(z) = p(x,y) + iq(x,y) = u(x,-y) - iv(x,-y)$ —— (5)

Equating real and imaginary part,
$p(x,y) = u(x,-y) \quad , q(x,y) = -v(x,-y)$

or $p(x,y) = u(x,\eta)$, $q(x,y) = -v(x,\eta)$
where $\eta = -y$ —— (6)

Now, since $f(x+i\eta)$ is analytic function of $x+\eta$ in D

$\therefore$ By Cauchy-Riemann equation,

$$u_x = v_\eta \quad , \quad u_\eta = -v_x \quad \text{—— (7)}$$

Now, $\because$ $p(x,y) = u(x,\eta)$

$$\therefore \quad p_x = \frac{\partial p}{\partial x} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial \eta}\cdot 0 \qquad \{\because \eta = -y\}$$

$$p_x = u_x$$

$$p_y = \frac{\partial p}{\partial y} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial u}{\partial x}\cdot 0 + \frac{\partial u}{\partial \eta}\cdot(-1)$$

$$= -\frac{\partial u}{\partial \eta}$$

$$p_y = -u_\eta$$

Similarly, $q(x,y) = -v(x,\eta)$

$$\therefore \quad q_x = \frac{\partial q}{\partial x} = -\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} - \frac{\partial v}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$= -\frac{\partial v}{\partial x}\cdot 1 - \frac{\partial v}{\partial \eta}\cdot 0$$

$$q_x = -v_x$$

and $q_y = \frac{\partial q}{\partial y} = -\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial y} - \frac{\partial v}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$

$= -\frac{\partial v}{\partial x}\cdot 0 - \frac{\partial v}{\partial \eta}\cdot(-1)$ $\{\because \eta = -y\}$

$$q_y = +v_\eta$$

Thus we get

$$\left.\begin{array}{ll} p_x = u_x, & p_y = -u_\eta \\ q_x = -v_x, & q_y = v_\eta \end{array}\right\} \text{—— (8)}$$

On combining eq^n (7) and (8), we get

$$p_x = q_y \text{ and } p_y = -q_x$$

This means that p and q satisfies the CR (Cauchy Riemann) eq^n and so $g(z)$ is analytic in $\mathcal{D}$.

Since $f(x)$ is real on a segment of real X-axis, therefore

$$f(z) = u(x,y) + iv(x,y)$$

$$\Rightarrow f(x) = u(x,0) + iv(x,0)$$

$$\Rightarrow f(x) = u(x,0) \text{ —— (9) } \{\because f(x) \text{ is real}\}$$

Now, $g(z) = p(x,y) + iq(x,y)$

$$\therefore g(x) = p(x,0) + iq(x,0)$$

$$\Rightarrow g(x) = u(x,0) - iv(x,0) \quad \{\text{from eq}^n \text{ (6)}\}$$

$$\Rightarrow g(x) = u(x,0) \text{ —— (10) } \{\because g(x) \text{ is real}\}$$

Thus, from (9) and (10), we get

$$g(x) = f(x)$$

$$\Rightarrow g(z) = f(z)$$

$$\Rightarrow \overline{f(\bar{z})} = f(z)$$
$$\Rightarrow f(\bar{z}) = \overline{f(z)}$$

Hence proved

To prove the converse, we assume that
$$\overline{f(\bar{z})} = f(z)$$

Then, we have
$$u(x,y) + iv(x,y) = u(x,-y) - iv(x,-y)$$

When this condition is satisfied on the segment AB then we have $y=0$ and so
$$u(x,0) + iv(x,0) = u(x,0) - iv(x,0)$$

On equating imaginary part
$$v(x,0) = -v(x,0)$$
$$\Rightarrow v(x,0) = 0$$

It follows that, $f(x) = u(x,0)$

Hence $f(x)$ is real on the part of the X-axis contained in the domain $\mathcal{D}$.

Analytic Continuation along a path

Germ:- Let (f, G) be a function element where G is a region and f is an analytic function in G. The germ of f at 'a' is the collection of all function element $(g, \mathcal{D})$ such that $a \in \mathcal{D}$ and $f(z) = g(z)$ $\forall$ z in a nbd of 'a' and is denoted by $[f]_a$

Note:- ① $[f]_a$ is a collection of function elements and it is not a function element itself.

② $(g, \mathcal{D}) \in [f]_a$ iff $(f, G) \in [g]_a$

Analytic Continuation along a path:-

Let $\gamma : [0,1] \to \mathbb{C}$ be a path and let there be a function element $(f_t, \mathcal{D}_t)$ for each t in $[0,1]$ such that

(a) $\gamma(t) \in \mathcal{D}_t$

(b) For each t in $[0,1]$, $\exists$ $\delta > 0$ such that $|s-t| < \delta \Rightarrow \gamma(s) \in \mathcal{D}_t$ and
$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$$

Then $(f_1, \mathcal{D}_1)$ ~~a~~ is said to be analytic continuation of $(f_0, \mathcal{D}_0)$ along the path γ.

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Proposition:- Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, \mathcal{D}_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Then
$$[f_1]_b = [g_1]_b$$

Proof:- In order to prove this result, we will show that the set $T = \{t \in [0,1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}$ is both open and closed in $[0,1]$

Since $0 \in T$ (by assumption, $[f_0]_a = [g_0]_a$)

$\therefore$ T is non-empty.

open — $\overset{\text{contains all}}{\text{interior point}}$
closed — contains all limit point

To show that T is open, fix t in T and assume that $t \neq 0$ or 1

If $t = 1$, then the proof is complete.
If $t=0$, then by the assumption $(a, a+\delta) \subset T$ for some $\delta > 0$

By the def^n of analytic continuation, there is a $\delta > 0$ such that

$$|s-t| < \delta \Rightarrow \gamma(s) \in \mathcal{D}_t \cap B_t$$

$$\left.\begin{array}{l} \text{and } [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)} \\ \text{and } [g_s]_{\gamma(s)} = [g_t]_{\gamma(s)} \end{array}\right\} \text{——} (1)$$

$\because t \in T$,
$\therefore f_t(z) = g_t(z) \quad \forall \ z \in \mathcal{D}_t \cap B_t$

Hence $[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)} \quad \forall \ \gamma(s) \in \mathcal{D}_t \cap B_t$

So from (1),

$$|s-t| < \delta \Rightarrow [f_s]_{\gamma(s)} = [g_s]_{\gamma(s)}$$

i.e. $(t-\delta, t+\delta) \subset T$ $\quad \{\because t \in (t-\delta, t+\delta)$ and t is arbitrary interior point$\}$
$\Rightarrow T$ is open

Next to show that T is closed.

Let t be a limit point of T.
Choose $\delta > 0$, so that $\gamma(s) \in \mathcal{D}_t \cap B_t$ and eq^n (1) is satisfied whenever $|s-t| < \delta$.

Since t is a limit point of T, therefore $\exists$ a point $s \in T$ with $|s-t| < \delta$, so that $\gamma(s) \in (\mathcal{D}_t \cap B_t) \cap (\mathcal{D}_s \cap B_s) = G$

$\Rightarrow$ G is non-empty

$\Rightarrow$ $f_s(z) = g_s(z) \quad \forall \; z \in G$ —— (2)

But by eq^n(1), we have

$$f_s(z) = f_t(z) \quad \text{and} \quad g_s(z) = g_t(z) \quad \forall z \in G \quad \text{—— (3)}$$

from (2) & (3), we have

$$f_t(z) = g_t(z) \quad \forall \; z \in G \quad \text{—— (4)}$$

Since G has a limit point in $\mathcal{D}_t \cap B_t$ so eq^n (4) gives,

$$[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}$$

$$\Rightarrow t \in T$$

Hence T is closed.

Lemma ① :- Let $\gamma : [0,1] \to \mathbb{C}$ be a path and let $\{(f_t, \mathcal{D}_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. For $0 \leq t \leq 1$, let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$ then either $R(t) \equiv \infty$ or $R : [0,1] \to (0, \infty)$ is continuous.

Proof:- If $R(t) \equiv \infty$, for some t then it is possible to extend f_t to an entire function,

$$f_s(z) = f_t(z) \quad \forall \; z \in \mathcal{D}_s$$

$$\Rightarrow R(s) \equiv \infty \quad \forall \; s \in [0,1]$$

$$\Rightarrow R(s) \equiv \infty$$

So we suppose that $R(t) < \infty \ \forall \ t \in [0,1]$ and we fix t in $[0,1]$ and let $\tau = \gamma(t)$

Let the power series expansion of f_t about τ z be:

$$f_t(z) = \sum_{n=0}^{\infty} a_n (z-\tau)^n$$

Now, let $\delta > 0$ be such that
$|s-t| < \delta \Rightarrow \gamma(s) \in D_t \cap B(\tau; R(t))$

And $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$

Again let us fix s with $|s-t| < \delta$, and $\sigma = \gamma(s)$

Now we can extend f_t to an analytic function in $B(\tau, R(t))$

Again, since f_s agree with f_t on a nbd of σ therefore f_s can be extended.

Consequently f_s is also analytic in $B(\tau, R(t)) \cup D_s$

Suppose f has the power series expansion about $z=\sigma$ as $f_s(z) = \sum_{n=0}^{\infty} b_n (z-\sigma)^n$

Then the radius of convergence $R(s)$ must be atleast as big as the distance from σ to the circle $|z-\tau| = R(t)$
i.e. $R(s) \geq d(\sigma, \Gamma) \geq R(t) - |\tau - \sigma|$
where $\Gamma = \{z: |z-\tau| = R(t)\}$

$\Rightarrow R(t) - R(s) \leq |\tau - \sigma|$

$\Rightarrow R(t) - R(s) \leq |\gamma(t) - \gamma(s)|$

and similarly $R(s) - R(t) \leq |\gamma(t) - \gamma(s)|$

$\therefore \max\{R(s) - R(t), -(R(s) - R(t))\} \leq |\gamma(t) - \gamma(s)|$

$\Rightarrow |R(s) - R(t)| \leq |\gamma(t) - \gamma(s)|$ for $|s-t| < \delta$,

Now since γ is continuous and defined in a compact domain $[0, 1]$.

$\therefore \gamma$ is uniformly continuous

Thus for a given $\epsilon > 0$, $\exists\ \delta > 0$ such that
$\gamma(s) \in \mathcal{D}_t \cap B(\tau, R(t))$

and f_s is analytic in $B(\tau, R(t)) \cup \mathcal{D}_s$

and $|s-t| < \delta \Rightarrow |\gamma(s) - \gamma(t)| < \epsilon$
$\Rightarrow |s-t| < \delta \Rightarrow |R(s) - R(t)| < \epsilon$

Hence R is uniformly continuous in the nbd $|s-t| < \delta$ of t.

$\therefore R$ must be continuous at t.

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Lemma②:- Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, \mathcal{D}_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. There is a number $\epsilon > 0$ such that if $\sigma : [0,1] \to \mathbb{C}$ is any path from a to b with $|\gamma(t) - \sigma(t)| < \epsilon$ for all t and if $\{(g_t, B_t) : 0 \leq t \leq 1\}$ is any continuation along σ with $[g_0]_a = [f_0]_a$ then $[g_1]_b = [f_1]_b$.

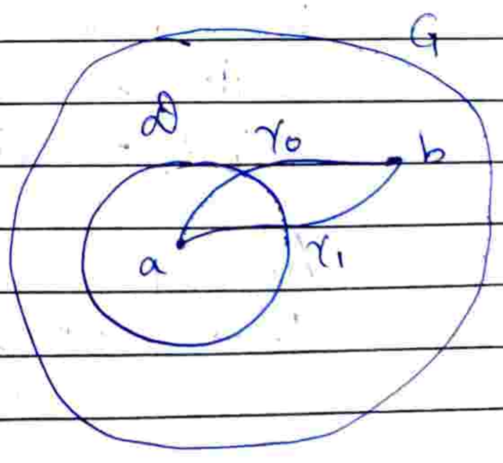
Def^n:- Let $(f, \mathcal{D})$ be a function element and let G be a region which contains $\mathcal{D}$ then $(f, \mathcal{D})$ admits unrestricted analytic continuation in G if for any path γ in G with initial point in $\mathcal{D}$ there is an analytic continuation of $(f, \mathcal{D})$ along γ.

** # Monodromy Theorem

Statement:- Let $(f, \mathcal{D})$ be a function element and let G be a region containing $\mathcal{D}$ such that $(f, \mathcal{D})$ admits unrestricted analytic continuation in G. Let $a \in \mathcal{D}$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b. Let $\{(f_t, \mathcal{D}_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuations of $(f, \mathcal{D})$ along γ_0 and γ_1 respectively. If γ_0 and γ_1 are fixed end point homotopic in G then $[f_1]_b = [g_1]_b$

Proof:- By hypothesis γ_0 and γ_1 are fixed end point homotopic in G, so there is a continuous function $\varphi : [0,1] \times [0,1] \to G$ such that

$\varphi(t, 0) = \gamma_0(t)$, $\varphi(t, 1) = \gamma_1(t)$
$\varphi(0, u) = a$, $\varphi(1, u) = b$
$\forall\ t, u \in [0,1]$.



Now, choose $u \in [0,1]$ and keep it fix.

Consider the path γ_u defined by

$$\gamma_u(t) = \varphi(t, u)$$

Again by the hypothesis, there is an analytic continuation $\{(h_{t,u}, D_{t,u}) : 0 \le t \le 1\}$ of (f, D) along the path γ_u.

Thus it follows that (by proposition).

$$[g_1]_b = [h_{1,1}]_b$$

and $[f_1]_b = [h_{1,0}]_b$

Hence it suffices to show that,

$$[h_{1,1}]_b = [h_{1,0}]_b$$

Define, $U = \{ u \in [0,1] : [h_{1,u}]_b = [h_{1,0}]_b \}$

Now, we will show that U is non-empty, open and closed subsets of $[0,1]$.

Since $0 \in U$, therefore $U \neq \phi$

Claim :- For $u \in [0,1]$, $\exists$ $\delta > 0$ such that

$$|u - v| < \delta \Rightarrow [h_{1,u}]_b = [h_{1,v}]_b \quad \text{--- (1)}$$

For a fixed $u \in [0,1]$, we can find $\epsilon > 0$ such that if σ is any path from a to b with $|\gamma_u(t) - \sigma(t)| < \epsilon$, $\forall$ t (By lemma (2))

And if $\{(k_t, E_t)\}$ is any analytic continuation of (f, D) along σ then,

$$[h_{1,u}]_b = [k_1]_b \quad \text{--- (2)}$$

Now, φ is uniformly continuous, $\therefore$ for any $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$|u-v| < \delta \Rightarrow |\varphi(t,u) - \varphi(t,v)| = |\gamma_u(t) - \gamma_v(t)| < \epsilon$$

Now, claim (1) follows by applying (2), i.e.,

$$|u-v| < \delta \Rightarrow [h_{1,u}]_b = [h_{1,v}]_b$$

Suppose $u \in U$ and let $\delta > 0$ be as in (1)

Then by def^n of U we have,

$$(u-\delta, u+\delta) \subset U$$

Hence U is open.

Let $u \in \overline{U}$ and $\delta > 0$ be chosen as in (1), then $\exists$ $v \in U$ such that

$$|u-v| < \delta$$

But by (1), $[h_{1,u}]_b = [h_{1,v}]_b$

and since $v \in U$,

$$[h_{1,v}]_b = [h_{1,0}]_b$$

It follows that, $[h_{1,u}]_b = [h_{1,0}]_b$

$$\Rightarrow u \in U$$

Hence U is closed.

This completes the proof of theorem.

22/02/20

Q. Show that the series:
(a) $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ and (b) $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$

are analytic continuations of each other.

Proof:- Given series are:- $f_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$$

The radius of convergence of the first power series is

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{1}{2^{n+1}}\right|^{1/n}$$

$$\Rightarrow R = \lim_{n\to\infty} 2^{\frac{n+1}{n}}$$

$$= \lim_{n\to\infty} 2^{1+\frac{1}{n}}$$

$$R = 2$$

Sum of the first power series is,

$$f_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right)$$

$$= \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right)$$

$$= \frac{1}{2-z} \quad \text{for } |z| < 2$$

Thus $f_1(z)$ is analytic within the circle $\Gamma_1 : |z| = 2$

Now the radius of convergence of the second series,

$$R = \lim_{n\to\infty} \left| (2-i)^{n+1} \right|^{1/n}$$

$$= \lim_{n\to\infty} \left| (2-i)^{1+\frac{1}{n}} \right|$$

$$= |2-i|$$

$$= \sqrt{5}$$

And sum of second series is,

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$$

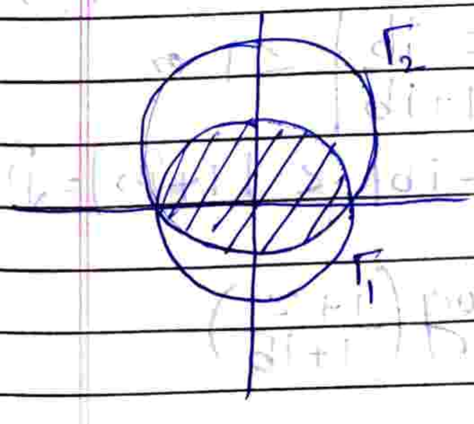
$$= \frac{1}{2-i} \sum_{n=0}^{\infty} \left(\frac{z-i}{2-i}\right)^n$$

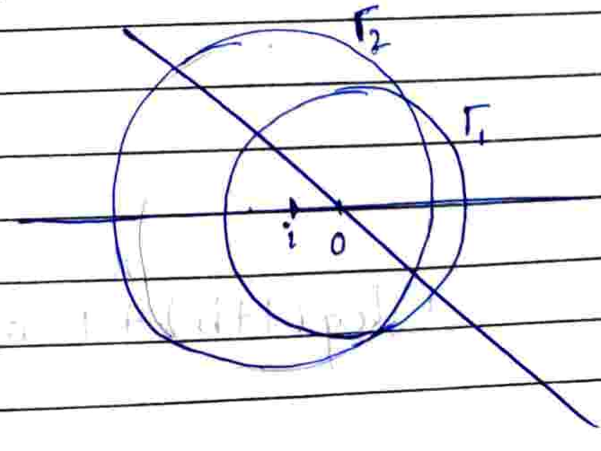
$$= \frac{1}{2-i} \left\{ \frac{1}{1-\left(\frac{z-i}{2-i}\right)} \right\}$$

$$= \frac{1}{2-i-z+i}$$

$$= \frac{1}{2-z} \quad \text{for } |z-i| < \sqrt{5}$$

Thus $f_2(z)$ is analytic within the circle $\Gamma_2 : |z-i| = \sqrt{5}$





From the above figure it is clear that the two power series represent the same function $\frac{1}{2-z}$

And $f_1(z) = f_2(z)$ in the common region of the circle $|z|=2$ and $|z-i| = \sqrt{5}$

Hence they are analytic continuation of each other.

*Q. Show that when $0 < b < 1$ the series:

$$\frac{1}{2}\log(1+b^2) + i\tan^{-1} b + \frac{z-ib}{1+ib} - \frac{1}{2}\frac{(z-ib)^2}{(1+ib)^2} + \cdots$$

is analytic continuation of the function defined by the series: $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$

Sol- The given series are:-

$$f_1(z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \cdots$$

$$\text{or } f_1(z) = \log(1+z) \quad \text{for } |z| < 1$$

Thus $f_1(z)$ is analytic within the circle $\Gamma_1 : |z| = 1$

$$f_2(z) = \frac{1}{2}\log(1+b^2) + i\tan^{-1} b + \frac{z-ib}{1+ib} - \frac{1}{2}\frac{(z-ib)^2}{(1+ib)^2} + \cdots$$

$$= \frac{1}{2}\log(1+b^2) + i\tan^{-1} b + \log\left[1 + \left(\frac{z-ib}{1+ib}\right)\right]$$

$$\text{for } \left|\frac{z-ib}{1+ib}\right| < 1 \text{ or}$$

$$|z-ib| < |1+ib| = \sqrt{1+b^2}$$

$$= \frac{1}{2}\log(1+b^2) + i\tan^{-1} b + \log\left(\frac{1+z}{1+ib}\right)$$

$$= \frac{1}{2}\log(1+b^2) + i\tan^{-1} b + \log(1+z) - \log(1+ib)$$

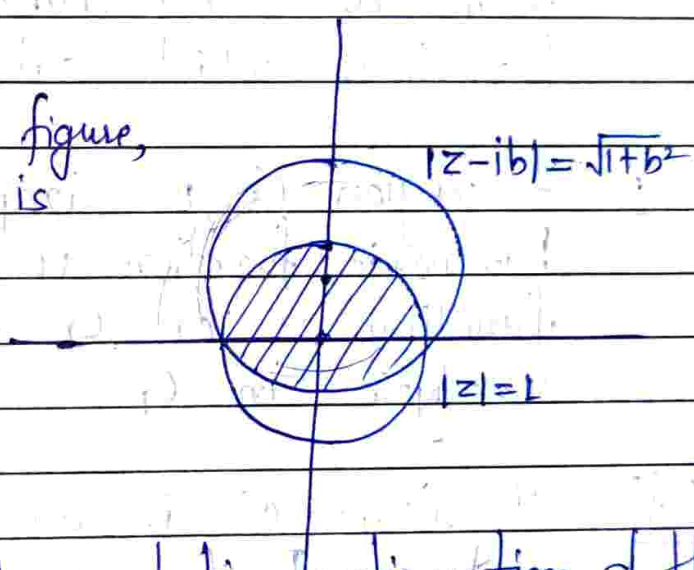
$$\left\{ \because \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} \right\}$$

$$= \log(1+ib) + \log(1+z) - \log(1+ib)$$

$$f_2(z) = \log(1+z)$$

Thus $f_2(z)$ is analytic for all values of z such that $|z-ib| < \sqrt{1+b^2}$

Thus from the adjacent figure, it is clear that there is a common region in which both $f_1(z)$ and $f_2(z)$ are analytic and $f_1(z) = f_2(z)$



Hence each function is analytic continuation of the other.

Unit-3

Harmonic Function

Def^n:- If G is an open subset of $\mathbb{C}$ then function $u: G \to \mathbb{R}$ is said to be harmonic if u has continuous second order partial derivatives and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Harmonic functions have the following properties:-

① A function f on a region G is analytic iff $\text{Re} f = u$ and $\text{Im} f = v$ are harmonic functions which satisfy the Cauchy Riemann equation

② A region G is simply connected iff for each harmonic function u on G, there is a harmonic function v on G such that $f = u + iv$ is analytic in G

Def^n:- If $f: G \to \mathbb{C}$ is an analytic function then $u = \text{Re} f$ and $v = \text{Im} f$ are called harmonic conjugates.

Note:- ① Every harmonic function in a simply connected region has a harmonic conjugate.

② If u is a harmonic function in G and D is a disc such that $D \subset G$ then there is a harmonic function v on D such that $u+iv$ is analytic in D.

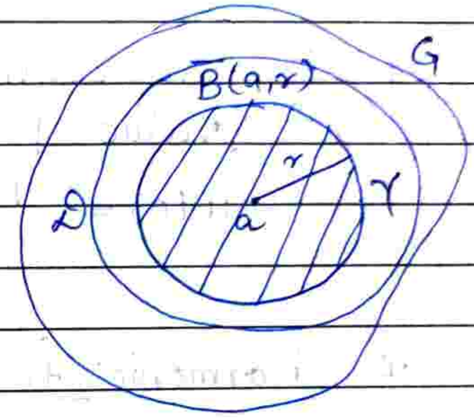
③ In other words, each harmonic function has harmonic conjugate locally

Mean Value Theorem

Statement:- Let $u: G \to \mathbb{R}$ be a harmonic function and $\overline{B}(a,r)$ be a closed disc such that $\overline{B}(a,r) \subset G$. If γ is the circle $|z-a| = r$ then,

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\, d\theta$$

Proof:- Suppose $\mathcal{D}$ is a disc such that,



$$\overline{B}(a,r) \subset \mathcal{D} \subset G$$

and $\gamma = \{ z : |z-a| = r \}$

And suppose that f is an analytic function in $\mathcal{D}$.

Applying the Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\, dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} \cdot rie^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\, d\theta \quad \text{——— (1)}$$

Suppose $u = \text{Re} f$, $v = \text{Im} f$

Then eq^n(1) can be written as,

$$u(a) + iv(a) = \frac{1}{2\pi}\int_0^{2\pi} \left[u(a+re^{i\theta}) + iv(a+re^{i\theta}) \right] d\theta$$

Equating real and imaginary parts, we obtain

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\, d\theta$$

$$v(a) = \frac{1}{2\pi}\int_0^{2\pi} v(a+re^{i\theta})\, d\theta$$

Hence proved

Define:- A continuous $u : G \to \mathbb{R}$ has the mean value property if whenever $\bar{B}(a,r) \subset G$

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\, d\theta.$$

Harmonic functions on a disc

Def^n:- (Poisson Kernel)

The function $P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$

where, $0 \le r < 1$ and $-\infty < \theta < \infty$ is called the Poisson kernel.

Proposition:- The Poisson kernel $P_r(\theta)$ can be expressed as $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2} = \text{Re}\left(\dfrac{1+re^{i\theta}}{1-re^{i\theta}}\right)$

Proof:- Consider the open unit disc $\{z : |z| < 1\}$ and suppose that $z = re^{i\theta}$, $0 \le r < 1$

Then,

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+z}{1-z} = (1+z)(1-z)^{-1}$$

$$= (1+z)(1+z+z^2+\ldots\ldots)$$

$$= 1 + z + z^2 + \cdots + z + z^2 + z^3 + \cdots$$

$$= 1 + 2 \sum_{n=1}^{\infty} z^n$$

$$= 1 + 2 \sum_{n=1}^{\infty} (re^{i\theta})^n$$

$$= 1 + 2 \sum_{n=1}^{\infty} r^n (\cos n\theta + i \sin n\theta)$$

$$\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = 1 + 2 \sum_{n=1}^{\infty} r^n \cos n\theta$$

$$= 1 + \sum_{n=1}^{\infty} r^n \left(e^{in\theta} + e^{-in\theta}\right)$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} \quad \left\{\begin{array}{l} e^{-in\theta} \to n\text{-neg.} \\ 1 \to n=0 \end{array}\right.$$

$$= P_r(\theta)$$

Again, $\dfrac{1+re^{i\theta}}{1-re^{i\theta}} = \dfrac{(1+re^{i\theta})}{(1-re^{i\theta})} \times \dfrac{(1-re^{-i\theta})}{(1-re^{-i\theta})}$

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1 - re^{-i\theta} + re^{i\theta} - r^2}{|1-re^{i\theta}|^2}$$

so that, $\operatorname{Re}\left(\dfrac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$

$\therefore$ $P_r(\theta) = \dfrac{1-r^2}{1-2r\cos\theta+r^2} = \operatorname{Re}\left(\dfrac{1+re^{i\theta}}{1-re^{i\theta}}\right)$

Hence proved

Proposition 2 :- The Poisson kernel $P_r(\theta)$ satisfies the following properties :

mcq ★ (a) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$

(b) $P_r(\theta) > 0$ for all θ,
$P_r(-\theta) = P_r(\theta)$
and P_r is periodic in θ with 2π.

(c) $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \leq \pi$
(Poisson kernel decreases as θ increases)

(d) for each $\delta > 0$, $\lim\limits_{r \to 1^-} P_r(\theta) = 0$

Proof (a) The Poisson kernel $P_r(\theta)$ is defined by the series given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta \quad \text{—(1)}$$

for $0 \leq r < 1$, $-\infty < \theta < \infty$

Since $|r^n \cos\theta| \leq r^n = M_n$ (say)

The infinite series $\sum\limits_{n=1}^{\infty} M_n = \sum\limits_{n=1}^{\infty} r^n$ is convergent.

Therefore, it follows that by Weiestrass M-test the series (1) is uniformly convergent in θ

Now, $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}\, d\theta$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \, d\theta$$

We know that, $\int_{-\pi}^{\pi} e^{in\theta} d\theta = \begin{cases} 2\pi, & n=0 \\ 0, & n \neq 0 \end{cases}$

$$\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta) \, d\theta = r^0 \cdot \frac{1}{2\pi} \cdot 2\pi = 1$$

(b) We have,

$$P_r(\theta) = Re\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) \quad \text{---(1)}$$

Again, $Re\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{|1-re^{i\theta}|^2}$

$$\Rightarrow P_r(\theta) = \frac{1-r^2}{|1-re^{i\theta}|^2} > 0$$

$$\Rightarrow P_r(\theta) > 0$$

Also, $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$

$$\therefore P_r(-\theta) = \frac{1-r^2}{1-2r\cos(-\theta)+r^2} = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

$$\Rightarrow P_r(-\theta) = P_r(\theta)$$

and $P_r(\theta+2\pi) = \frac{1-r^2}{1-2r\cos(\theta+2\pi)+r^2}$

$$= \frac{1-r^2}{1-2r\cos\theta+r^2}$$

$$= P_r(\theta)$$

Hence P_r is of 2π periodic.

(c) $P_r(\theta) < P_r(\delta)$ if $0 < \delta < |\theta| \leq \pi$

Let $0 < \delta < |\theta| \leq \pi$
and define $f : [\delta, \theta] \to \mathbb{R}$ by
$$f(t) = P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}$$

Then, $f'(t) < 0$ for $\delta \leq t \leq \theta$

It follows that the function is monotone decreasing in $[\delta, \theta]$, so $f(\delta) > f(\theta)$

$\Rightarrow P_r(\theta) < P_r(\delta)$

(d) We have to show that,
$$\lim_{r\to 1^-} \left[\sup \{ P_r(\theta) : \delta \leq |\theta| \leq \pi \}\right] = 0$$

According to the property (c),
$P_r(\theta) \leq P_r(\delta)$ if $\delta \leq |\theta| \leq \pi$

So, it suffices to show that $\lim_{r\to 1^-} P_r(\delta) = 0$

Now,
$$\lim_{r\to 1^-} P_r(\delta) = \lim_{r\to 1} \frac{1-r^2}{1-2r\cos\delta + r^2}$$
$$= 0$$
for $\delta < |\theta| \leq \pi$

Hence proved

24/02/20.

Harnack's Inequality

Statement :- If $u : \overline{B}(a, R) \to \mathbb{R}$ is continuous, harmonic function and $u \geq 0$ then for $0 \leq r < R$ and all θ,

$$\frac{R-r}{R+r}\, u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r}\, u(a)$$

Proof :- The Poisson Kernel $P_r(\theta)$ can be expressed as,

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} \quad \text{—— (1)}$$

If $R > 0$ then put $\frac{r}{R}$ for r in (1) and denotind $P_{r/R}(\theta)$ by $P(R, r, \theta)$, then we get

$$P(R, r, \theta) = \frac{1-\left(\frac{r}{R}\right)^2}{1-2\frac{r}{R}\cos\theta + \left(\frac{r}{R}\right)^2}$$

$$= \frac{R^2 - r^2}{R^2 - 2rR\cos\theta + r^2} \quad \text{—— (2)}$$

for $0 \leq r < R$ & all θ

~~This~~

Now, we know the result " If $u : \overline{D} \to \mathbb{R}$ is a continuous function that is harmonic in D, then

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, u(e^{it})\, dt$$

for $0 \leq r < 1$ and all θ ".

Hence if u is continuous on $\overline{B}(a, R)$ and harmonic in $B(a, R)$, then

$$u(a+re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}\, u(a+Re^{it})\,dt \quad \text{--- (3)}$$

{Using eqn (2)}

Modify eqn (2) as,

$$P(R, r, \theta-t) = \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}$$

$$= \frac{R^2-r^2}{|Re^{it}-re^{i\theta}|^2} \quad \text{--- (4)}$$

Moreover,

$$R-r \leq |Re^{it}-re^{i\theta}| \leq R+r$$

$$\Rightarrow (R-r)^2 \leq |Re^{it}-re^{i\theta}|^2 \leq (R+r)^2$$

$$\Rightarrow (R-r)^2 \leq R^2-2rR\cos(\theta-t)+r^2 \leq (R+r)^2$$

$$\Rightarrow \frac{1}{(R-r)^2} \geq \frac{1}{R^2-2rR\cos(\theta-t)+r^2} \geq \frac{1}{(R+r)^2}$$

$$\Rightarrow \frac{(R+r)(R-r)}{(R+r)^2} \leq \frac{(R+r)(R-r)}{R^2-2rR\cos(\theta-t)+r^2} \leq \frac{(R+r)(R-r)}{(R-r)^2}$$

$$\Rightarrow \frac{R-r}{R+r} \leq \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} \leq \frac{R+r}{R-r} \quad \text{--- (5)}$$

If $u \geq 0$, then inequality (5) yields,

$$\frac{R-r}{R+r}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+Re^{it})\,dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2}\cdot u(a+Re^{it})\,dt$$

$$\leq \frac{R+r}{R-r}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+Re^{it})\,dt$$

Since u is 2π period, so we can write

$$\frac{R-r}{R+r}\frac{1}{2\pi}\int_0^{2\pi} u(a+Re^{it})dt \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} u(a+Re^{it})dt$$

$$\leq \frac{R+r}{R-r}\frac{1}{2\pi}\int_0^{2\pi} u(a+Re^{it})dt$$

By mean value theorem, $u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a+Re^{it})dt$

$$\therefore \frac{R-r}{R+r} u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r} u(a)$$

(from eq^n(3))

Hence proved.

Def^n :- If G is an open subset of $\mathbb{C}$, then $Har(G)$ is the space of harmonic functions on G. Since $Har(G) \subset C(G, \mathbb{R})$, it is a metric space that it inherits from $C(G, \mathbb{R})$

Harnack's Theorem

Statement :- Let G be a region.

(a) The metric space $Har(G)$ is complete.

(b) If $\{u_n\}$ is a sequence in $Har(G)$ such that $u_1 \leq u_2 \leq \cdots\cdots$ then either $u_n(z) \to \infty$ uniformly on compact subsets of G or $\{u_n\}$ converges in $Har(G)$ to a harmonic function.

Proof :- (a) We know that $C(G, \mathbb{R})$ is a complete metric space

And a closed subspace of a complete metric space is complete.

Thus to show that $Har(G)$ is complete, it is sufficient to show that it is a closed subspace of $C(G, \mathbb{R})$

So let $\{u_n\}$ be a sequence in $Har(G)$ such that $u_n \to u$ in $C(G, \mathbb{R})$

Then it follows that u has the mean value property and so u must be harmonic.

(b) Without loss of generality we may assume that $u_1 \geq 0$

Let $u(z) = \sup\{u_n(z) : n \geq 1\}$
so for each $z \in G$, there are two possibilities:
i) $u(z) \equiv \infty$ or
ii) $u(z) \in \mathbb{R}$
and $u_n(z) \to u(z)$

Now define two subsets A and B of G as follows:
$A = \{z \in G \mid u(z) = \infty\}$
$B = \{z \in G \mid u(z) < \infty\}$

Then clearly, $G = A \cup B$ and $A \cap B = \phi$
($\because G$ is connected)

Claim: A and B are open

Verification :- If $a \in G$
Let R be so choosen that
$\overline{B}(a, R) \subset G$

$\therefore$ By Harnack's inequality,

$$\frac{R-|z-a|}{R+|z-a|}\, u_n(a) \le u_n(z) \le \frac{R+|z-a|}{R-|z-a|}\, u_n(a) \quad \text{—— (1)}$$

Case I :- If $a \in A$

then, $u(a) = \infty$

i.e. $u_n(a) \to \infty$

So, the left half of (1) gives that $u_n(z) \to \infty$, $\forall\, z \in B(a, R)$

$\Rightarrow B(a,R) \subset A$

$\Rightarrow a \in B(a,R) \subset A$

$\Rightarrow A$ is a nbd of a

$\Rightarrow A$ is a nbd of each of its points

Hence A is open.

Case II :- If $a \in B$

then $u(a) < \infty$

i.e., $u_n(a) < \infty$

$\therefore$ the right half of (1) gives that $u_n(z) < \infty$, $\forall\, z \in B(a, R)$

$\Rightarrow B(a,R) \subset B$

$\Rightarrow a \in B(a,R) \subset B$

$\Rightarrow B$ is a nbd of a

$\Rightarrow B$ is a nbd of each of its points

Hence B is open.

Now, since G is connected and $G = A \cup B$, $A \cap B = \phi$

$\therefore$ either $A = G$ or $B = G$.

Case I :- Suppose $A = G$

i.e. $u(z) = \infty \quad \forall \; z \in G$

Again, if $\overline{B}(a, R) \subset G$ and $0 < \rho < R$

then,

$$\frac{R-\rho}{R+\rho} > 0$$

$\therefore$ from the left side of (1),

$$\frac{R-\rho}{R+\rho} u_n(a) \leq u_n(z) \quad \text{for } |z-a| \leq \rho$$

$\Rightarrow M u_n(a) \leq u_n(z)$, for $|z-a| \leq \rho$ and $M = \dfrac{R-\rho}{R+\rho}$

Hence $u_n(z) \to \infty$ uniformly for $z \in \overline{B}(a, \rho)$

$\Rightarrow u_n(z) \to \infty$ uniformly for z in any compact subset of G.

Case II :- Suppose $B = G$

then $u(z) < \infty \quad \forall \; z \in G$.

If $\rho < R$, then (say)

$$M = \frac{R-\rho}{R+\rho} > 0 \quad \text{and} \quad N = \frac{R+\rho}{R-\rho} > 0$$

Thus, $M u_n(a) \leq u_n(z) \leq N u_n(a)$, for $|z-a| \leq \rho$

Now, if ~~$M \leq n$~~ $m \leq n$, then

$$0 \leq u_n(z) - u_m(z)$$

$$0 \leq N u_n(a) - M u_m(a) \quad \left\{ \begin{array}{l} \because u_n(z) \leq N u_n(a) \\ -u_m(z) \leq -M u_m(a) \end{array} \right\}$$

$$\leq C\left[u_n(a) - u_m(a)\right]$$

for some constant C

$\Rightarrow \{u_n(z)\}$ is uniformly Cauchy's sequence on $\overline{B}(a,\rho)$

$\Rightarrow \{u_n\}$ is a Cauchy sequence in $Har(G)$
$\Rightarrow \{u_n\}$ must converge to a harmonic function in $Har(G)$

Hence proved

27.2.20

Subharmonic and superharmonic function

Let G be a region and let $\varphi: G \to \mathbb{R}$ be a continuous function. Then φ is said to be subharmonic function if whenever $\overline{B}(a,r) \subset G$,

$$\varphi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{i\theta})\, d\theta$$

and is said to be superharmonic if whenever $\overline{B}(a,r) \subset G$,

$$\varphi(a) \ge \frac{1}{2\pi}\int_0^{2\pi} \varphi(a+re^{i\theta})\, d\theta$$

Note: ① φ is superharmonic iff $-\varphi$ is subharmonic

② Every harmonic function is subharmonic as well as superharmonic.

③ u is harmonic iff u is both subharmonic and superharmonic.

④ If $a_1, a_2 \ge 0$ and ~~θ_1, θ~~ φ_1, φ_2 are subharmonic then $a_1\varphi_1 + a_2\varphi_2$ is also subharmonic

Perron family :

Let G be a region and $f: \partial_\infty G \to \mathbb{R}$ be a continuous function. Then the Perron family $\mathcal{P}(f, G)$, consist of all subharmonic functions $\varphi: G \to \mathbb{R}$ such that

$$\limsup_{z \to a} \varphi(z) \le f(a) \quad , \forall \ a \in \partial_\infty G$$

Perron family associated with f

Let G be a region and $f: \partial_\infty G \to \mathbb{R}$ be a continuous function. Then

$$u(z) = \sup \{\varphi(z) : \varphi : \mathcal{P}(f, G)\}$$

defines a harmonic function u in G.

The harmonic function u is called the Perron function associated with f.

Dirichlet Problem

A region G is called a Dirichlet region if the Dirichlet problem can be solved for G, i.e., G is Dirichlet region if for each continuous function $f: \partial_\infty G \to \mathbb{R}$, there is a continuous function $u: \overline{G} \to \mathbb{R}$ such that u is harmonic in G and $u(z) = f(z)$, $\forall \ z \in \partial_\infty(G)$

Note:- For a set G and a point $a \in \partial_\infty G$, let
$G(a, r) = G \cap B(a, r) \quad \forall \ r > 0$

Barrier

Let G be a region and let $a \in \partial_\infty G$. A barrier for G at a is a family $\{\psi_r : r>0\}$ of functions such that

(a) ψ_r defined and superharmonic on $G(a,r)$ with $0 \leq \psi_r(z) \leq 1$

(b) $\lim_{z \to a} \psi_r(z) = 0$

(c) $\lim_{z \to w} \psi_r(z) = 1$, for w in $G \cap \{w : |w-a| = r\}$

Define $\hat{\psi}_r$ by letting

(i) $\hat{\psi}_r(z) = \psi_r(z)$ for $z \in G(a,r)$
(ii) $\hat{\psi}_r(z) = 1$ for $z \in G - B(a,r)$.

We have the following observations:-

(1) $\hat{\psi}_r$ is superharmonic, so the function $\hat{\psi}$ approach the function which is everywhere but 0 at $z=a$

(2) If G is a Dirichlet region then there is a barrier for G at each point of $\partial_\infty G$

Theorem:- Let G be a region and let $a \in \partial_\infty G$ such that there is a barrier for G at a. If $f : \partial_\infty G \to \mathbb{R}$ is continuous and u is the Perron function associated with f then $\lim_{z \to a} u(z) = f(a)$

Proof: Suppose $\{\psi_r : r>0\}$ is a barrier for G at a and assume that $a \neq \infty$ itself.

We also assume that $f(a) = 0$

Let $\epsilon > 0$ be given. Then $\exists\ \delta > 0$ such that
$\omega \in \partial_\infty G$ and $|\omega - a| < 2\delta \Rightarrow |f(\omega) - f(a)| < \epsilon$
$\Rightarrow |f(\omega)| < \epsilon$

Let $\psi = \psi_\delta$. Then we define $\hat{\psi} : G \to \mathbb{R}$ by,
$$\hat{\psi}(z) = \psi(z) \quad \text{for } z \in G(a, \delta)$$
$$\text{and } \hat{\psi}(z) = 1 \quad \text{for } z \in G - B(a, \delta)$$

Then $\hat{\psi}$ is superharmonic

further, if $|f(\omega)| \leq M,\ \forall\ \omega \in \partial_\infty G$
Then, $-M\hat{\psi} - \epsilon$ is subharmonic

Claim:- $-M\hat{\psi} - \epsilon \in \mathcal{P}(f, G)$

If $\omega \in \partial_\infty G - B(a, \delta)$, then
$$\limsup_{z \to \omega} [-M\hat{\psi}(z) - \epsilon] = -M - \epsilon < f(\omega)$$

Since $\hat{\psi}(z) \geq 0$,
$$\therefore \limsup_{z \to \omega} [-M\hat{\psi}(z) - \epsilon] \leq -\epsilon \quad \forall\ \omega \in \partial_\infty G$$

Hence $-M\hat{\psi} - \epsilon \in \mathcal{P}(f, G)$

Hence, $-M\hat{\psi} - \epsilon \leq u(z) \quad \forall\ z \in G$ —— (1)

Similarly, $\liminf_{z \to \omega} [M\hat{\psi}(z) + \epsilon] \geq \limsup_{z \to \omega} \varphi(z)$
$\forall\ \varphi \in \mathcal{P}(f, G)$

By the maximum principle,

$$\varphi(z) \leq M\hat{\varphi}(z) + \varepsilon \quad \forall \varphi \in \mathcal{P}(f, G) \quad \text{--- (2)}$$

Combining (1) & (2), we get

$$-M\hat{\varphi}(z) - \varepsilon \leq u(z) \leq M\hat{\varphi}(z) + \varepsilon \quad \text{--- (3)} \quad \forall z \in G$$

But $\lim_{z \to a} \hat{\varphi}(z) = \lim_{z \to a} \varphi(z) = 0$

Since ε was arbitrary, so (3) gives

$$-M \lim_{z \to a} \hat{\varphi}(z) \leq \lim_{z \to a} u(z) \leq M \lim_{z \to a} \hat{\varphi}(z)$$

$$\Rightarrow \quad 0 \leq \lim_{z \to a} u(z) \leq 0$$

$$\Rightarrow \quad \lim_{z \to a} u(z) = 0 = f(a)$$

Hence proved.

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Green's Function

Let G be a region in the complex plane and let $a \in G$. A Green's function of G with singularity at 'a' is a function $g_a : G \to \mathbb{R}$ with the following properties :-

(a) g_a is harmonic in $G - \{a\}$

(b) $G(z) = g_a(z) + \log|z-a|$ is harmonic in a disc about a.

(c) $\lim_{z \to w} g_a(z) = 0$, for each w in $\partial_\infty G$

Note:- ① For a given region G and a point $a \in G$, g_a need not exist.

However if it exists, it is unique.

② We also observe that Green's function is positive.

<u>Theorem</u>:- Let G be a bounded Dirichlet's region. Then for each $a \in G$, there is a Green's function on G with singularity at 'a'.

<u>Proof</u>:- Define $f : \partial G \to \mathbb{R}$ by $f(z) = \log|z-a|$ and let $u : \overline{G} \to \mathbb{R}$ be the unique continuous function which is harmonic in G such that

$$u(z) = f(z) \quad , \forall \ z \in \partial G$$

Then $g_a(z) = u(z) - \log|z-a|$ is a function having the following properties,

(a) g_a is harmonic in G with singularity at 'a', i.e, g_a is harmonic in $G - \{a\}$

(b)
$$G(z) = g_a(z) + \log|z-a|$$
$$= (u(z) - \log|z-a|) + \log|z-a|$$
$$= u(z)$$

This is harmonic in G, i.e. harmonic in a disc about a.

(c)
$$\lim_{z \to w} g_a(z) = \lim_{z \to w} [u(z) - \log|z-a|]$$
$$= \lim_{z \to w} [f(z) - \log|z-a|]$$
$$= \lim_{z \to w} [\log|z-a| - \log|z-a|]$$
$$= 0 \qquad \forall \ w \in \partial_\infty G$$

$\partial_\infty \Omega = \overline{\Omega}$

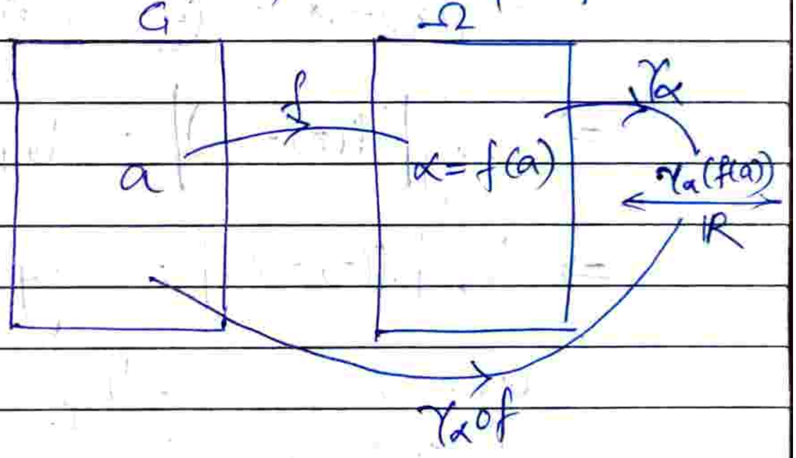
Hence $g_a(z)$ is the required Green's function on G with singularity at 'a'.

Theorem:- Let G and Ω be regions such that there is a one-one analytic function f of G onto Ω. Let $a \in G$ and $\alpha = f(a)$. If g_a & γ_α are Green's functions for G and Ω with singularity 'a' and 'α' respectively, then

$$g_a(z) = \gamma_\alpha(f(z))$$

Proof:- Let $\varphi : G \to \mathbb{R}$ be defined by $\varphi = \gamma_\alpha \circ f$

In order to show that $\varphi = g_a$, it is suffices to show that φ has the properties of Green's function with singularity at $z = a$



Since $f : G \to \Omega$ is one-one and analytic and γ_α is harmonic with singularity at α and $\alpha = f(a)$

Therefore, $\varphi = \gamma_\alpha \circ f$ is harmonic with singularity at a.

Now, If $w \in \partial_\infty G$ and if it can be shown that $\lim_{n\to\infty} \varphi(z_n) = 0$ for any sequence $\{z_n\}$ in G with $z_n \to w$ then $\lim_{z \to w} \varphi(z) = 0$ will follow.

Now, $\{f(z_n)\}$ is a sequence in Ω and so there is a subsequence $\{z_{n_k}\}$ such that $f(z_{n_k}) \to \omega$ in $\overline{\Omega}$ in $\mathbb{C}_\infty$

Thus $\gamma_\alpha(f(z_{n_k})) \to 0$ as $k \to \infty$ for any convergent subsequence of $\{f(z_n)\}$, it follows that

$$\lim_{n\to\infty} \varphi(z_n) = \lim_{n\to\infty} \gamma_\alpha(f(z_n)) = 0$$

Hence, $\lim_{z\to w} \varphi(z) = 0$

We now consider the power series expansion of f about $z=a$,

$$f(z) = \alpha + A_1(z-a) + A_2(z-a)^2 + \cdots\cdots$$

$$\Rightarrow f(z) - \alpha = (z-a)\left[A_1 + A_2(z-a) + \cdots\cdots\right]$$

$$\Rightarrow \log|(f(z)-\alpha)| = \log|z-a| + \log|A_1 + A_2(z-a) + \cdots|$$

$$\Rightarrow \log|(f(z)-\alpha)| = \log|z-a| + h(z) \quad \text{---(1)}$$

where $h(z) = \log|A_1 + A_2(z-a) + \cdots\cdots|$ is harmonic near $z=a$, since $A_1 \neq 0$

Suppose $\gamma_\alpha(w) = \Delta(w) - \log|w-\alpha|$
where Δ is a harmonic function on Ω.

Now from (1), we have

$$\varphi(z) = (\gamma_\alpha \circ f)(z)$$
$$= \gamma_\alpha(f(z))$$

$$= \Delta(f(z)) - \log|f(z)-\alpha|$$
$$= (\Delta f(z)) - h(z)) - \log|z-a| \quad \{\text{from } 1\}$$

$$= (\Delta \circ f - h)(z) - \log|z-a|$$

Thus, $\varphi(z) + \log|z-a| = (\Delta \circ f - h)(z)$

Since $(\Delta \circ f - h)$ is harmonic near $z=a$, it follows that $\varphi(z) + \log|z-a|$ is harmonic near $z=a$.

Hence φ is also a Green's function with singularity at 'a' and having the properties same as g_a.

Hence $\boxed{g_a(z) = \gamma_\alpha(f(z))}$

Hence proved.

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Unit-4

Canonical Product

By Weierstrass factorisation theorem, it is well-known that " There exist an entire function with arbitrary prescribed zeros z_n provided that in the case of infinitely many zeros $z_n \to \infty$
Every entire function with these and no other zeros can be written in the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right) \exp\left\{\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \cdots + \frac{1}{p_n}\left(\frac{z}{z_n}\right)^{p_n}\right\} \quad \text{---(1)}$$

The proof of Weierstrass factorization theorem has shown that the product

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right) \exp\left\{\frac{z}{z_n} + \frac{1}{2}\left(\frac{z}{z_n}\right)^2 + \cdots + \frac{1}{h}\left(\frac{z}{z_n}\right)^{h}\right\} \quad \text{---(2)}$$

converges and represent an entire function provided the series $\sum_{n=1}^{\infty} \frac{1}{h+1}\left(\frac{R}{|z_n|}\right)^{h+1}$ converges for all R i.e. $\sum \frac{1}{|z_n|^{h+1}} < \infty$

Assume that h is smallest integer for which this series converges.

The expression (2) is called the canonical product associated with the sequence z_n and h is called the genus of the canonical product.

For example, entire function of genus 0 is of the form $cz^m \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)$ with $\sum_{n=1}^{\infty}\frac{1}{|z_n|} < \infty$

The canonical representation of an entire function of genus 1 is either of the form $cz^m \prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right) \exp\left\{\frac{z}{z_n}\right\}$ with $\sum_{n=1}^{\infty}\frac{1}{|z_n|^2} < \infty$

or $\sum_{n=1}^{\infty}\frac{1}{|z_n|} = \infty$

Example:- Construct the canonical product associated with sequence of negative integers

Sol- The sequence of zeros of the required entire function is $\{z_n\}_{n=1}^{\infty} = \{-n\}_{n=1}^{\infty}$

Thus the genus ~~p~~ of the canonical product is clearly 1. Since $h=1$ is the smallest integer such that $\sum_{n=1}^{\infty}\frac{1}{|z_n|^{h+1}} = \sum_{n=1}^{\infty}\frac{1}{|-n|^{1+1}} < \infty$

It follows that the canonical product associated with the sequence $\{-n\}_{n=1}^{\infty}$ of negative integers is

$$f(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

Clearly $f(z)$ is an entire function with simple zeros at $z=-1,-2,-3,\ldots$

Example:- Write the entire function $\sin z$ and $\cos z$ as canonical product.

Sol- Consider the function $f(z)=\sin\pi z$

$\pi z = n\pi$

Thus the zeros of $f(z)$ are: $z=n$, $n=0,\pm1,\pm2,\ldots$

Now, we have

$$\sum_{n=1}^{\infty}\frac{1}{|z_n|^{h+1}} = \sum\frac{1}{|n|^{h+1}}$$

We observe that the genus of the canonical product is 1.

Also $z=0$ is a simple zero

Hence we obtain a representation of the form

$$\sin\pi z = ze^{g(z)}\prod_{n=-\infty}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n} \quad \text{——— (1)}$$

Note that here n takes all integral values except 0

Now to determine $g(z)$ we take log on both side of (1),

$$\log \sin \pi z = \log z + g(z) + \sum_{n=-\infty}^{\infty}\left[\log\left(1 - \frac{z}{n}\right) + \frac{z}{n}\right] \quad \text{--- (2)}$$

Differentiating wrt z

$$\frac{\pi \cos \pi z}{\sin \pi z} = \frac{1}{z} + g'(z) + \sum_{n=-\infty}^{\infty}\left[\frac{-1/n}{1 - \frac{z}{n}} + \frac{1}{n}\right]$$

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n=-\infty}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right] \quad \text{--- (3)}$$

But, $\pi \cot \pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right)$ --- (4)

On comparing (3) & (4), we get

$$g'(z) = 0$$

Thus $g(z)$ is a constant.

Also we have,

$$\lim_{z \to 0} \frac{\sin \pi z}{z} = \pi$$

Thus from (1),

$$\lim_{z \to 0} \frac{\sin \pi z}{z} = \lim_{z \to 0} e^{g(z)} \prod_{n=-\infty}^{\infty}\left(1 - \frac{z}{n}\right) e^{z/n}$$

$$\pi = e^{g(z)}$$

Hence,

$$\sin \pi z = \pi z \prod_{n=-\infty}^{\infty}\left(1 - \frac{z}{n}\right) e^{z/n}$$

$$\Rightarrow \sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) \cancel{e^{z/n}}$$

$$\Rightarrow \quad \sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

This the canonical product for $\sin z$.

Jensen's Inequality

Let $f(z)$ be an entire function with $f(0) \neq 0$. Also let $r_1, r_2, \dots, r_n$ be the moduli of zeros $z_1, z_2, \dots, z_n$ of $f(z)$, arranged as non-decreasing sequence, multiple zero being repeated. Then

$$R^n |f(0)| \leq M(R)\, r_1 \cdot r_2 \cdots r_n \quad \text{if } r_n < R < r_{n+1}$$

where $M(R)$ denote the maximum modulus of $|f(z)|$ on the circle $|z| = R$

<u>Proof</u>:- Consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \left(\frac{R^2 - z\bar{z}_i}{R(z - z_i)}\right) \quad \text{—— (1)}$$

Since $f(z)$ is an entire function,

$\therefore$ $F(z)$ is also an entire function.

On the circle $|z| = R$, for all i, $1 \leq i \leq n$, we have

$$R^2(z - z_i)(\bar{z} - \bar{z}_i) = R^2\left[z\bar{z} - z\bar{z}_i - z_i\bar{z} + z_i\bar{z}_i\right]$$

$$= R^2\left[z\bar{z} - (z\bar{z}_i + z_i\bar{z}) + z_i\bar{z}_i\right]$$

$$= R^2\left[R^2 - (z\bar{z}_i + z_i\bar{z}) + z_i\bar{z}_i\right]$$

$$\left\{\because |z|^2 = R^2 \Rightarrow z\bar{z} = R^2\right\}$$

$$= R^4 - R^2(z\bar{z}_i) - R^2(z_i\bar{z}) + (z\bar{z})(z_i\bar{z}_i)$$

$$\left\{\because R^2 = z\bar{z}\right\}$$

$$= R^2(R^2 - z\bar{z}_i) - z_i\bar{z}(R^2 - z\bar{z}_i)$$
$$= (R^2 - z\bar{z}_i)(R^2 - z_i\bar{z})$$

$$R(z-z_i)\,\overline{R(z-z_i)} = (R^2 - z\bar{z}_i)\overline{(R^2 - \bar{z}_i z)}$$

$$\Rightarrow |R(z-z_i)|^2 = |R^2 - z\bar{z}_i|^2$$

$$\Rightarrow \left|\frac{R^2 - z\bar{z}_i}{R(z-z_i)}\right|^2 = 1$$

$$\Rightarrow \left|\frac{R^2 - z\bar{z}_i}{R(z-z_i)}\right| = 1 \quad \text{--- (2)}$$

From eq^n (1) and (2),

$$|F(z)| = f(z) \quad \text{on} \quad |z| = R \quad \text{--- (3)}$$

By maximum modulus principle, we have

$$|F(z)| \leq \max_{|z|=R} |F(z)|$$
$$= \max_{|z|=R} |f(z)| \quad (\text{by (3)})$$

Let $M(R)$ denote the maximum modulus of $|f(z)|$ on the circle $|z| = R$, we obtain

$$|F(z)| \leq M(R)$$

Now, putting $z = 0$ in (1), we get

$$F(0) = f(0) \prod_{i=1}^{n} \left(\frac{R^2 - 0}{R(-z_i)}\right)$$

$$\Rightarrow \left| f(0) \prod_{i=1}^{n} \left(\frac{R^2}{R(-z_i)} \right) \right| = |F(0)| \leq M(R)$$

$$\Rightarrow |f(0)| \prod_{i=1}^{n} \frac{R}{|z_i|} \leq M(R)$$

$$\Rightarrow |f(0)| \frac{R \cdot R \ldots\ldots (n \text{ times})}{|z_1| \cdot |z_2| \ldots\ldots |z_n|} \leq M(R)$$

$$\Rightarrow \frac{R^n |f(0)|}{r_1 \cdot r_2 \ldots\ldots r_n} \leq M(R)$$

$$\Rightarrow R^n |f(0)| \leq M(R) |z_1 \cdot z_2 \ldots\ldots z_n|$$

This completes the proof.

Jensen's Formula

Statement :- Let $f(z)$ be analytic in the closed disc $|z| \leq R$. Assume that $f(0) \neq 0$ and no zeros of $f(z)$ lie on $|z| = R$. If $z_1, z_2, \ldots, z_n$ are zeros of $f(z)$ in the open disc $|z| < R$, each repeated as often as its multiplicity, then

$$\log |f(0)| = -\sum_{i=1}^{n} \log \left(\frac{R}{|z_i|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| \, d\phi$$

Proof:- Consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \quad \text{--- (1)}$$

Since $f(z)$ is analytic in $|z| \leq R$,

$\therefore$ $F(z)$ also analytic in $|z| \leq R$.

Also $F(z) \neq 0$ on $|z| \leq R$

Hence $F(z)$ is analytic and never zero on an open disc $|z| < \rho$ for some $\rho > R$

Now, we write $z = Re^{i\phi}$, then

$$\left| \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \right| = \left| \prod_{i=1}^{n} \frac{R^2 - \bar{z}_i Re^{i\phi}}{R^2 e^{i\phi} - Rz_i} \right|$$

$$= \prod_{i=1}^{n} \left| \frac{R(R - \bar{z}_i e^{i\phi})}{Re^{i\phi}(R - z_i e^{-i\phi})} \right|$$

$$= \prod_{i=1}^{n} \left| \frac{R - \bar{z}_i e^{i\phi}}{R - z_i e^{-i\phi}} \right| \quad \left[\because |e^{i\phi}| = 1\right]$$

$$= 1 \quad \text{--- (2)}$$

$\left\{ \because R - \bar{z}_i e^{i\phi} \text{ and } R - z_i e^{-i\phi} \text{ are conjugates \& it is known that the moduli of conjugates are same} \right\}$

Hence from eq^n (1) & (2), we get

$$|F(z)| = |f(z)| \quad \text{on } |z| = R$$

Now, since $F(z)$ is analytic and non-zero in $|z| < \rho$, $\therefore$ the function $\log F(z)$ is harmonic in $|z| < \rho$

So, by Gauss Mean Value theorem for $\log|F(z)|$, we get

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(Re^{i\phi})| \, d\phi \quad \text{--- (3)}$$

Now by (1),

$$F(0) = f(0) \prod_{i=1}^{n} \left(\frac{-R}{z_i} \right)$$

$$\Rightarrow |F(0)| = |f(0)| \prod_{i=1}^{n} \frac{R}{|z_i|}$$

$$\Rightarrow \log|F(0)| = \log|f(0)| + \sum_{i=1}^{n} \frac{R}{|z_i|} \quad \text{—— (4)}$$

From (3) and (4), we have

$$\log|f(0)| + \sum_{i=1}^{n} \frac{R}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\phi})|\,d\phi \quad \text{—— (5)}$$

Also, since $|F(z)| = |f(z)|$ on $|z| = R$

$$\therefore |F(Re^{i\phi})| = |f(Re^{i\phi})| \quad \text{—— (6)}$$

Thus from (5),

$$\log|f(0)| + \sum_{i=1}^{n} \frac{R}{|z_i|} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\,d\phi$$

$$\Rightarrow \log|f(0)| = -\sum_{i=1}^{n} \frac{R}{|z_i|} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\phi})|\,d\phi.$$

This completes the proof.

Jensen's Theorem

Statement:- Let $f(z)$ be analytic for $|z| \leq R$ and let $r_1, \dots, r_n$ be the moduli of the zeros of $f(z)$ in $|z| < R$ arranged as a non-decreasing sequence. Then if $r_n \leq r < r_{n+1}$, prove that

$$\log \frac{r^n |f(0)|}{r_1 . r_2 \dots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, d\theta$$

<u>Proof</u>: First we consider the function

$$F(z) = f(z) \prod_{i=1}^{n} \left(\frac{R^2 - z\bar{z}_i}{R(z - z_i)} \right) \quad \text{——— (1)}$$

The construction $F(z)$ shows that $F(z)$ is free from singularity.

Moreover, $F(z)$ has no zeros within and on the circle $\gamma : |z| = R$. It follows that $F(z)$ is analytic within and on the circle γ.

Hence, by Cauchy's Integral formula, we have

$$\log F(0) = \frac{1}{2\pi i} \int_{\gamma} \frac{\log F(z)}{z - 0} \, dz \quad \text{——— (2)}$$

Take $z = Re^{i\theta}$ on γ

so that $dz = iRe^{i\theta} d\theta$

From (2), we have

$$\log F(0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\log F(Re^{i\theta})}{Re^{i\theta}} Rie^{i\theta} \, d\theta$$

$$\Rightarrow \log F(0) = \frac{1}{2\pi} \int_0^{2\pi} \log F(Re^{i\theta}) \, d\theta \quad \text{——— (3)}$$

Equating real parts on both sides, we get

$$\log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(Re^{i\theta})| \, d\theta \quad \text{——— (4)}$$

Now, we write $z = Re^{i\phi}$, then

$$\left|\prod_{i=1}^{n} \frac{R^2 - z\bar{z}_i}{R(z-z_i)}\right| = \left|\prod_{i=1}^{n} \frac{R^2 - (Re^{i\phi})\bar{z}_i}{R(Re^{i\phi} - z_i)}\right|$$

$$= \left|\prod_{i=1}^{n} \frac{R(R - e^{i\phi}\bar{z}_i)}{Re^{i\phi}(R - e^{-i\phi}z_i)}\right|$$

$$= \left|\prod_{i=1}^{n} \frac{(R - e^{i\phi}\bar{z}_i)}{e^{i\phi}(R - e^{-i\phi}z_i)}\right|$$

$$= \prod_{i=1}^{n} \frac{|R - e^{i\phi}\bar{z}_i|}{|R - e^{-i\phi}z_i|} \quad \left\{\because |e^{i\phi}| = 1\right\}$$

$$= 1 \quad \left\{\because |R - e^{i\phi}\bar{z}_i| = |R - e^{-i\phi}z_i|\right\}$$

$\therefore$ from ①, $|F(z)| = |f(z)|$

$$\Longrightarrow |F(Re^{i\theta})| = |f(Re^{i\theta})|$$

So by eqⁿ ④, we have

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta \quad — ⑤$$

Put $z=0$ in ①, we have

$$|F(0)| = |f(0)| \left|\prod_{i=1}^{n} \frac{R^2}{R(-z_i)}\right|$$

$$|F(0)| = |f(0)| \prod_{i=1}^{n} \frac{R}{|z_i|}$$

Taking log both sides, we get

$$\log|F(0)| = \log\left[|f(0)| \prod_{i=1}^{n} \frac{R}{|z_i|}\right]$$

$$\Rightarrow \log|F(0)| = \log\left(|f(0)| \frac{R.R.R.\ldots.R\ (n \text{ times})}{|z_1|.|z_2|\ldots.|z_n|}\right)$$

$$= \log\left(|f(0)| \frac{R^n}{|z_1|\ldots|z_n|}\right)$$

$$= \log \frac{|f(0)|\ R^n}{r_1.r_2\ldots r_n} \quad \text{——— (6)}$$

From (5) and (6),

$$\log\left(\frac{|f(0)|\ R^n}{r_1.r_2\ldots,r_n}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

Replacing R by r, then

$$\log \frac{r^n |f(0)|}{r_1.r_2.\ldots,r_n} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

Proved

Poisson Jensen Formula

Statement:- Let $f(z)$ be analytic in the closed disc $|z| \le R$. Assume that $f(0) \neq 0$ and no zeros of $f(z)$ lies on $|z| = R$. If $z_1, z_2, \ldots, z_n$ are the zeros of $f(z)$ in the open disc $|z| < R$ each repeated as often as its multiplicity and $z = re^{i\theta}$, $0 \le r < R$, $f(z) \neq 0$, then

$$\log|f(z)| = -\sum_{i=1}^{n} \log\left|\frac{R^2 - \bar{z}_i z}{R(z - z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\phi})|\,d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

Proof:- Consider the function,

$$F(z) = f(z) \prod_{i=1}^{n} \frac{R^2 - \overline{z}_i z}{R(z - z_i)} \quad \text{——(1)}$$

The construction of $F(z)$ shows that $F(z)$ is free from singularity, i.e., $F(z)$ is analytic in any domain in which $f(z)$ is analytic and $F(z) \neq 0$ for $|z| \leq R$

Hence, $F(z)$ is analytic and never zero on an open disc $|z| < \rho$, for some $\rho > R$

We write $z = Re^{i\phi}$ and so

$$\left| \prod_{i=1}^{n} \frac{R^2 - \overline{z}_i z}{R(z - z_i)} \right| = \left| \prod_{i=1}^{n} \frac{R^2 - \overline{z}_i Re^{i\phi}}{R(Re^{i\phi} - z_i)} \right|$$

$$= \left| \prod_{i=1}^{n} \frac{R(R - \overline{z}_i e^{i\phi})}{Re^{i\phi}(R - z_i e^{-i\phi})} \right|$$

$$= \prod_{i=1}^{n} \frac{|R - \overline{z}_i e^{i\phi}|}{|R - z_i e^{-i\phi}|} \qquad \{\because |e^{i\phi}| = 1\}$$

$$= 1 \qquad \{\because |R - \overline{z}_i e^{i\phi}| = |R - z_i e^{-i\phi}|\}$$

From (1),

$$|F(z)| = |f(z)| \quad \text{——(2)}$$

Since $F(z)$ is analytic and non-zero on an open disc $|z| < \rho$, therefore $\log F(z)$ is analytic in $|z| < \rho$ and so real part $\log|F(z)|$ is harmonic there.

Hence, by Poisson's formula for $\log|F(z)|$, we get

$$\log|F(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\log|F(Re^{i\phi})|}{R^2-2rR\cos(\theta-\phi)+r^2}\,d\phi \quad \text{--- (3)}$$

We have, $|F(Re^{i\phi})| = |f(Re^{i\phi})|$ --- (4)

From (1), taking log

$$\log|F(z)| = \log|f(z)| + \sum_{i=1}^{n}\log\left|\frac{R^2-\overline{z_i}z}{R(z-z_i)}\right| \quad \text{--- (5)}$$

From eq's (3), (4) and (5), we have

$$\log|f(z)| + \sum_{i=1}^{n}\log\left|\frac{R^2-\overline{z_i}z}{R(z-z_i)}\right| = \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\log|f(Re^{i\phi})|}{R^2-2rR\cos(\theta-\phi)+r^2}\,d\phi$$

$$\Rightarrow \log|f(z)| = -\sum_{i=1}^{n}\log\left|\frac{R^2-\overline{z_i}z}{R(z-z_i)}\right| + \frac{1}{2\pi}\int_0^{2\pi}\frac{(R^2-r^2)\log|f(Re^{i\phi})|}{R^2-2rR\cos(\theta-\phi)+r^2}\,d\phi.$$

Hence proved.

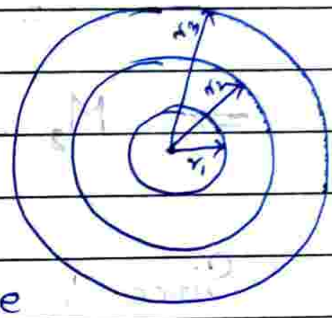
Hadamard's Three Circles Theorem

Statement:- Let $f(z)$ be analytic in the annular region $r_1 \leq |z| \leq r_3$ and $r_1 < r_2 < r_3$. If M_1, M_2, M_3 be the max $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$ respectively, then

$$M_2^{\log(r_3/r_1)} \leq M_3^{\log(r_2/r_1)} M_1^{\log(r_3/r_2)}$$

Proof:- Let $F(z) = z^k f(z)$, where k is a real constant.

Then $F(z)$ is analytic in the annulus $r_1 \leq |z| \leq r_3$



If k is not an integer, then $F(z)$ is a multivalued function and so one can choose the principal branch.

For this, we cut the annulus along the negative real axis and we obtain a domain in which the principal branch of this function is analytic.

By the maximum modulus principle, the maximum value of $|F(z)|$ is obtained on the boundary of the cut annulus.

Now consider a branch of this function which is analytic in the part of the annulus for which $\frac{\pi}{2} \leq \arg z \leq \frac{3\pi}{2}$, we see that the principal value cannot attain its maximum modulus on the cut

Hence the max $|F(z)|$ occurs on one of the bounding circles.

Thus it is shown that when $r_1 \le |z| \le r_3$,

$$|F(z)| \le \max\{r_1^k M_1, r_3^k M_3\}. \quad \text{--- (1)}$$

Hence, on $|z| = r_2$, we must have

$$r_2^k M_2 \le \max\{r_1^k M_1, r_3^k M_3\}$$

$$\Rightarrow r_2^k M_2 \le r_1^k M_1$$

$$\Rightarrow M_2 \le \left(\frac{r_1}{r_2}\right)^k M_1$$

$$\Rightarrow M_2 \le \left(\frac{r_2}{r_1}\right)^{-k} M_1 \quad \text{--- (2)}$$

Since k is at our choice,

$\therefore$ k is defined by the equation

$$r_1^k M_1 = r_3^k M_3$$

$$\Rightarrow \log(r_1^k M_1) = \log(r_3^k M_3)$$

$$\Rightarrow k \log r_1 + \log M_1 = k \log r_3 + \log M_3$$

$$\Rightarrow k \log r_1 - k \log r_3 = \log M_3 - \log M_1$$

$$\Rightarrow k \log\left(\frac{r_1}{r_3}\right) = \log\left(\frac{M_3}{M_1}\right)$$

$$\Rightarrow k = \frac{\log\left(\frac{M_3}{M_1}\right)}{\log\left(\frac{r_1}{r_3}\right)}$$

$$\Rightarrow k = \frac{-\log(M_3/M_1)}{\log(r_3/r_1)} \quad \text{--- (3)}$$

Putting value of k on eq^n ②,

$$M_2 \leq \left(\frac{r_2}{r_1}\right)^{\frac{\log(M_3/M_1)}{\log(r_3/r_1)}} . M_1$$

$$\Rightarrow M_2^{\log(r_3/r_1)} \leq \left(\frac{r_2}{r_1}\right)^{\log(M_3/M_1)} . M_1^{\log(r_3/r_1)}$$

$$\Rightarrow M_2^{\log(r_3/r_1)} \leq \left(\frac{M_3}{M_1}\right)^{\log(r_2/r_1)} . M_1^{\log(r_3/r_1)}$$

$$\left\{\because a^{\log b} = b^{\log a}\right\}$$

$$\Rightarrow M_2^{\log(r_3/r_1)} \leq M_3^{\log(r_2/r_1)} . M_1^{-\log(r_2/r_1)} . M_1^{\log(r_3/r_1)}$$

$$\Rightarrow M_2^{\log(r_3/r_1)} \leq M_3^{\log(r_2/r_1)} M_1^{\log\left(\frac{r_3/r_1}{r_2/r_1}\right)}$$

$$\Rightarrow M_2^{\log(r_3/r_1)} \leq M_3^{\log(r_2/r_1)} . M_1^{\log(r_3/r_2)}$$

Proved

Note :- $a^{\log b} = b^{\log a}$

Let $a^{\log b} = x$ —①

$$\Rightarrow \log a^{\log b} = \log x$$

$$\Rightarrow (\log b) \log a = \log x$$

$$\Rightarrow \log a . \log b = \log x$$

$$\Rightarrow \log b^{\log a} = \log x$$

$$\Rightarrow \cancel{\log}\, b^{\log a} = x \quad \text{—②}$$

From ① & ②, $a^{\log b} = b^{\log a}$.

Order of an Entire function

Let $f(z)$ be an entire function. Denote by $M(r)$ the maximum modulus of $f(z)$ on the circle $|z|=r$, i.e., $M(r) = \max\{|f(z)| : |z|=r\}$

Proposition I :- If $f(z)$ is an entire transcedental function with maximum modulus $M(r)$, then

$$\lim_{r\to\infty} \frac{\log M(r)}{\log r} = \infty$$

Proof:- Suppose if possible,

$$\lim_{r\to\infty} \inf \frac{\log M(r)}{\log r} = \mu < \infty$$

Then for any given $\epsilon > 0$, there exists an increasing sequence $\{r_n\}$ converging to ∞ such that

$$\frac{\log M(r_n)}{\log r_n} < \mu + \epsilon$$

$$\Rightarrow \log M(r_n) < (\mu+\epsilon) \log r_n$$

$$\Rightarrow \log M(r_n) < \log r_n^{(\mu+\epsilon)}$$

$$\Rightarrow M(r_n) < r_n^{(\mu+\epsilon)} \quad \forall\, r_n$$

Thus by Cauchy's Inequality, we have

$$|a_n| \leq \frac{M(r_n)}{r_n^{\lambda}} < \frac{r_n^{(\mu+\epsilon)}}{r_n^{\lambda}} = r_n^{\mu+\epsilon-\lambda}$$

Taking r_n as sufficiently large. It follows that $a_n = 0$ for all $\lambda > \mu+\epsilon$

Hence $f(z)$ is a polynomial of degree not higher than $[\mu]$.

Note:- By this result, we conclude that an entire transcedental function may be utilized to measure the growth of an entire function

★ An entire function $f(z)$ is said to be of finite order a defined by

$$\lambda = \inf\{a : M(r) \leq \exp(r^a) \text{ for all sufficiently large } r\}$$

If, for sufficiently large values of r,

$$M(r) > \exp(r^a)$$

then $f(z)$ is said to be of infinite order.

For example, e^z is of finite order (of order 1) while e^{e^z} is of infinite order.

Theorem:- Let f be a non-constant function. Define

$$\rho_1 = \inf\{\lambda \geq 0 : M(r) \leq \exp(r^\lambda) \text{ for sufficiently large } r\}$$

$$\rho_2 = \lim_{r\to\infty} \sup \frac{\log\log M(r)}{\log r}$$

Then $\rho_1 = \rho_2$

Proof:- Let f be a non-constant function.

Define $\rho_1 = \inf\{\lambda \geq 0 : M(r) \leq \exp(r^\lambda) \text{ for sufficiently large } r\}$ — ①

$$\rho_2 = \lim_{r\to\infty} \sup \frac{\log\log M(r)}{\log r} \quad \text{— ②}$$

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If $\rho_1 < \infty$

Then for any given $\epsilon > 0$, $\exists$ a number $R(\epsilon) > 0$ such that,

$$M(r) < \exp(r^{\rho_1+\epsilon}) \quad \forall\ r > R(\epsilon)$$

$$\left\{ \because \rho_1 + \epsilon > \lambda \;\; \Rightarrow \lambda < \rho_1 + \epsilon \right. \quad \left. (\text{from eq}^n \text{ 1}) \right\}$$

$$\Rightarrow \log M(r) < \log \exp(r^{\rho_1+\epsilon})$$

$$\Rightarrow \log M(r) < r^{\rho_1+\epsilon}$$

$$\Rightarrow \log \log M(r) < \log r^{\rho_1+\epsilon}$$

$$\Rightarrow \log\log M(r) < (\rho_1+\epsilon)\log r$$

$$\Rightarrow \frac{\log\log M(r)}{\log r} < \rho_1 + \epsilon$$

$$\Rightarrow \rho_2 \leq \rho_1 + \epsilon \qquad (\text{from } 2)$$

Letting $\epsilon \to 0$, we get

$$\rho_2 \leq \rho_1 \quad \text{——— } 3$$

If $\rho_2 < \infty$

then for any given $\epsilon > 0$,

$$\frac{\log\log M(r)}{\log r} < \rho_2 + \epsilon \qquad (\text{from } 2)$$

$$\Rightarrow \log\log M(r) < (\rho_2+\epsilon)\log r$$

$$\Rightarrow \log\log M(r) < \log r^{(\rho_2+\epsilon)}$$

$$\Rightarrow \log M(r) < r^{\rho_2+\epsilon}$$

$$\Rightarrow M(r) < \exp(r^{\rho_2+\epsilon})$$

$$\Rightarrow \inf\{\lambda \geq 0 : M(r) \leq \exp(r^{\lambda}) \text{ for sufficiently large } r\} < \rho_2 + \epsilon$$

$$\Rightarrow \rho_1 < \rho_2 + \epsilon \qquad \{\text{from } 1\}$$

letting $\epsilon \to 0$, we get

$$\rho_1 \leq \rho_2 \quad \text{——} (4)$$

From eqn (3) & (4), we get

$$\rho_1 = \rho_2$$

This completes the proof.

<u>Theorem</u>:- If the real part of an entire function $g(z)$ satisfies the inequality $\operatorname{Re} g(z) < r^{\rho+\epsilon}$ for every $\epsilon > 0$ & all sufficiently large r, then $g(z)$ is a polynomial of degree not exceeding ρ.

<u>Proof</u>:- Let $g(z)$ be an entire function satisfies the inequality

$$\operatorname{Re} g(z) < r^{\rho+\epsilon} \quad \text{——} (1)$$

for every $\epsilon > 0$ and all sufficiently large r.

By taylor's expansion we have,

$$g(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{n+1}}\, dz$ —— (2)

C being the circle $|z| = r$

Now, when $n > 0$, we have

$$\frac{1}{2\pi i}\int_C \frac{\overline{g(z)}}{z^{n+1}}\, dz = \frac{1}{2\pi i}\int_C \left(\sum_{m=0}^{\infty} \overline{a_m}\, \overline{z^m}\right) \frac{dz}{z^{n+1}}$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty} \overline{a_m} \int_C \frac{\overline{z^m}}{z^{n+1}}\, dz \quad \text{——} (3)$$

Since $|z| = r$, therefore $z = re^{i\theta}$ or $\overline{z} = re^{-i\theta}$

$\Rightarrow \quad dz = re^{i\theta} i\, d\theta$

Putting $z = re^{i\theta}$ in eq^n ③,

$$\frac{1}{2\pi i}\int_C \frac{\overline{g(z)}}{z^{n+1}}\,dz = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\overline{a_m}\int_0^{2\pi}\frac{\overline{(re^{i\theta})^m}}{(re^{i\theta})^{n+1}}\,(re^{i\theta} i\, d\theta)$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty}\overline{a_m}\int_0^{2\pi}\frac{r^m e^{-mi\theta}}{r^n e^{ni\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty}\overline{a_m}\int_0^{2\pi} r^{m-n}\, e^{-(n+m)i\theta}\,d\theta$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty}\overline{a_m}\, r^{m-n}\left[\frac{e^{-(m+n)i\theta}}{-(m+n)i}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty}\overline{a_m}\, r^{m-n}\cdot 0$$

$$= 0$$

i.e. $\displaystyle \frac{1}{2\pi i}\int_C \frac{\overline{g(z)}}{z^{n+1}}\,dz = 0$ ——— ④

Adding eq^n ② & ④, we get

$$a_n + 0 = \frac{1}{2\pi i}\int_C \frac{g(z)+\overline{g(z)}}{z^{n+1}}\,dz$$

$$\Rightarrow \quad a_n = \frac{1}{2\pi i}\int_C \frac{2\,\mathrm{Re}\, g(z)}{z^{n+1}}\,dz$$

By putting $z = re^{i\theta}$,

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi}\frac{2\,\mathrm{Re}\, g(re^{i\theta})}{(re^{i\theta})^{n+1}}\, re^{i\theta} i\, d\theta$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{Re\, g(re^{i\theta})}{(re^{i\theta})^n} d\theta$$

$$\Rightarrow |a_n| = \left| \frac{1}{\pi}\int_0^{2\pi} \frac{Re\, g(re^{i\theta})}{r^n e^{ni\theta}} d\theta \right|$$

$$\Rightarrow |a_n| \leq \frac{1}{\pi}\int_0^{2\pi} \frac{|Re\, g(re^{i\theta})|}{r^n |e^{ni\theta}|} d\theta$$

$$\Rightarrow |a_n| \leq \frac{1}{\pi r^n}\int_0^{2\pi} |Re\, g(re^{i\theta})|\, d\theta$$

$$\Rightarrow r^n|a_n| \leq \frac{1}{\pi}\int_0^{2\pi} |Re\, g(re^{i\theta})|\, d\theta \quad \text{——— (5)}$$

Again from eq^n (2), by putting $n=0$, we get

$$a_0 = \frac{1}{2\pi i}\int_C \frac{g(z)}{z} dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(re^{i\theta})}{re^{i\theta}} re^{i\theta}\, i d\theta$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\, d\theta$$

$$\Rightarrow Re\, a_0 = \frac{1}{2\pi}\int_0^{2\pi} Re\, g(re^{i\theta})\, d\theta$$

$$\Rightarrow 2 Re\, a_0 = \frac{1}{\pi}\int_0^{2\pi} Re\, g(re^{i\theta})\, d\theta \quad \text{——— (6)}$$

Adding eq^n (5) & (6), we get

$$r^n |a_n| + 2\,\mathrm{Re}\, a_0 \le \frac{1}{\pi}\int_0^{2\pi} \left[\left|\mathrm{Re}\, g(re^{i\theta})\right| + \mathrm{Re}\, g(re^{i\theta}) \right] d\theta$$

$$\Rightarrow r^n |a_n| + 2\,\mathrm{Re}\, a_0 \le \frac{1}{\pi}\int_0^{2\pi} \left\{ |\mathrm{Re}\, g| + \mathrm{Re}\, g \right\} d\theta$$

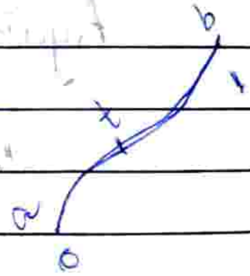
$$\Rightarrow r^n |a_n| + 2\,\mathrm{Re}\, a_0 \le \frac{1}{\pi}\int_0^{2\pi} 2\,\mathrm{Re}\, g \, d\theta$$

{$\because$ But the integrand is equal to $2\,\mathrm{Re}\, g$ or 0 according as $\mathrm{Re}\, g >$ or ≤ 0 }

By assumption, $\mathrm{Re}\, g < r^{\rho+\epsilon}$

$$\therefore r^n |a_n| + 2\,\mathrm{Re}\, a_0 < \frac{1}{\pi}\int_0^{2\pi} 2r^{\rho+\epsilon}\, d\theta$$

$$< \frac{2}{\pi}\int_0^{2\pi} r^{\rho+\epsilon}\, d\theta$$



$$\le \frac{2}{\pi} r^{\rho+\epsilon}\, 2\pi$$

$$\Rightarrow r^n |a_n| + 2\,\mathrm{Re}\, a_0 < 4r^{\rho+\epsilon}$$

$$\Rightarrow r^n |a_n| < 4r^{\rho+\epsilon} - 2\,\mathrm{Re}\, a_0$$

$$\Rightarrow |a_n| < 4r^{\rho+\epsilon-n} - 2\,\mathrm{Re}\, a_0 r^{-n}$$

letting $r \to \infty$ then, $|a_n| \le 0$

so, $a_n = 0$ when $n > \rho$

Hence $g(z)$ is a polynomial of degree not exceeding ρ.

This completes the proof.

Example:- Find the order of polynomial

$$p(z) = a_0 + a_1 z + \dots + a_n z^n \quad , \quad a_n \neq 0$$

Sol- The given polynomial is:

$$p(z) = a_0 + a_1 z + \dots\dots + a_n z^n \quad , \quad a_n \neq 0$$

$\because$ The maximum modulus of $p(z)$ is given by

$$M(r) = |a_n z^n| = |a_n| r^n \quad , \text{ for large } |z| = r$$

so that

$$\lim_{r \to \infty} \frac{\log \log M(r)}{\log r} = \lim_{r \to \infty} \frac{\log \log (|a_n| r^n)}{\log r}$$

$$\left\{ \frac{\infty}{\infty} \text{ form} \right\}$$

$$= \lim_{r \to \infty} \frac{\frac{1}{\log(|a_n| r^n)} \cdot \frac{1}{|a_n| r^n} \cdot |a_n| n r^{n-1}}{1/r}$$

$$= \lim_{r \to \infty} \frac{n}{\log(|a_n| r^n)}$$

$$= 0 \quad \left\{ \because \log(|a_n| r^n) \to \infty \text{ as } r \to \infty \right\}$$

Hence the order of the polynomial $p(z)$ is zero.

Example 2:- Find the order of following functions

(i) e^{az}, $a \neq 0$ (ii) e^{z^λ}, λ a positive integer

(iii) e^{e^z}

Sol-(i) The given function is e^{az}, $a \neq 0$

The maximum modulus, $M(r) = e^{|a||z|} = e^{|a| \cdot r}$

so that $\lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\log\log (e^{|a|r})}{\log r}$

$= \lim_{r\to\infty} \frac{\log(|a|r)}{\log r}$ $\left\{\frac{\infty}{\infty}\text{ form}\right\}$

$= \lim_{r\to\infty} \frac{\frac{1}{|a|.r}.|a|}{1/r}$

$= 1$

Hence order of e^{az} is 1.

(ii) The given function is e^{z^λ}, λ a positive integer.

Here $M(r) = e^{|z|^\lambda} = e^{r^\lambda}$

$\therefore \lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\log\log(e^{r^\lambda})}{\log r}$

$= \lim_{r\to\infty} \frac{\log(r^\lambda)}{\log r}$ $\left\{\frac{\infty}{\infty}\text{ form}\right\}$

$= \lim_{r\to\infty} \frac{\lambda\log r}{\log r}$

$= \lambda$

Hence the order of e^{z^λ} is λ

(iii) Given function is e^{e^z}.

Here, $M(r) = e^{e^{|z|}} = e^{e^r}$

$$\therefore \lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \lim_{r\to\infty} \frac{\log\log e^{e^r}}{\log r}$$

$$= \lim_{r\to\infty} \frac{\log e^r}{\log r}$$

$$= \lim_{r\to\infty} \frac{r}{\log r} \quad \left\{\frac{\infty}{\infty} \text{ form}\right\}$$

$$= \lim_{r\to\infty} \frac{1}{1/r}$$

$$= \lim_{r\to\infty} r$$

$$= \infty$$

Hence e^{e^z} is of infinite order.

Example 3:- Find the order of the following functions:-

i) $\cos z$ ii) $\cos\sqrt{z}$ iii) $\sin z$

Sol- i) Given function is $\cos z$.

We have the series expansion of $\cos z$ as:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

We conclude that,

$$|\cos z| \leq 1 + \frac{|z|^2}{2!} + \frac{|z|^4}{4!} + \frac{|z|^6}{6!} + \cdots$$

$$\leq 1 + \frac{r^2}{2!} + \frac{r^4}{4!} + \frac{r^6}{6!} + \cdots \quad \{\because |z| \leq r\}$$

$$= \frac{1}{2}(e^r + e^{-r}) \quad \text{in } |z| = r$$

Thus $|\cos z| \leq \frac{e^r + e^{-r}}{2}$ if $|z| \leq r$

$\therefore M(r) = \frac{e^r + e^{-r}}{2}$

$\Rightarrow M(r) = e^r \left(\frac{1 + e^{-2r}}{2} \right)$

so that

$$\log M(r) = \log e^r + \log \left(\frac{1 + e^{-2r}}{2} \right)$$

$$= r + \log \left(\frac{1 + e^{-2r}}{2} \right)$$

$$= r \left[1 + \frac{1}{r} \log \left(\frac{1 + e^{-2r}}{2} \right) \right]$$

Hence,

$$\lim_{r \to \infty} \frac{\log \log M(r)}{\log r} = \lim_{r \to \infty} \frac{\log \left[r \left\{ 1 + \frac{1}{r} \log \left(\frac{1 + e^{-2r}}{2} \right) \right\} \right]}{\log r}$$

$$= \lim_{r \to \infty} \frac{\log r + \log \left(1 + \frac{1}{r} \log \left(\frac{1 + e^{-2r}}{2} \right) \right)}{\log r}$$

$$= \lim_{r \to \infty} \left[1 + \frac{\log \left(1 + \frac{1}{r} \log \left(\frac{1 + e^{-2r}}{2} \right) \right)}{\log r} \right]$$

$= 1$, as the second term tends to 0 as $r \to \infty$

Hence the order of $\cos z$ is 1.

Exponents of convergence (σ)

Let $\{z_n\}$ be a sequence of non-zero complex numbers such that $|z_1| \le |z_2| \le \dots \le |z_n| \to \infty$ as $n \to \infty$
The exponents of convergence σ of the sequence is defined by
$$\sigma = \inf\left\{\alpha > 0 : \sum_{n=1}^{\infty} \frac{1}{|z_n|^{\alpha}} < \infty\right\}$$

Note:- 1. If the sequence is finite, then $\sigma = 0$

2. $\sigma = \infty$ iff $\sum_{n=1}^{\infty} |z_n|^{-\alpha} = \infty$, $\forall\ \alpha > 0$

3. $\sigma = 0$ iff $\sum_{n=1}^{\infty} |z_n|^{-\alpha} < \infty$, $\forall\ \alpha > 0$

Proposition:- The convergence of exponent σ of a sequence $\{z_n\}$ is given by
$$\sigma = \limsup_{n \to \infty} \frac{\log n}{\log |z_n|}$$

Proof:- Suppose σ is finite.
Then the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\alpha}}$ converges for every $\alpha > \sigma$.

Since the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\alpha}}$ converges, it follows that $\limsup_{n \to \infty} \frac{n}{|z_n|^{\alpha}} = 0$ (n^{th} partial sum)

Therefore for $\epsilon = 1$, $\exists\ n_0 \in \mathbb{N}$ such that

$$\left| \frac{n}{|z_n|^\alpha} - 0 \right| < 1 \quad \forall\ n > n_0$$

$\Rightarrow \dfrac{n}{|z_n|^\alpha} < 1, \quad \forall\ n > n_0$

$\Rightarrow \log n - \log |z_n|^\alpha < 0 \quad \forall\ n > n_0$

$\Rightarrow \log n - \alpha \log |z_n| < 0, \quad \forall\ n > n_0$

$\Rightarrow \log n < \alpha \log |z_n|, \quad \forall\ n > n_0$

$\Rightarrow \alpha \log |z_n| > \log n, \quad \forall\ n > n_0$

$\Rightarrow \alpha > \dfrac{\log n}{\log |z_n|}, \quad \forall\ n > n_0$

$\Rightarrow \alpha \geq \limsup\limits_{n \to \infty} \dfrac{\log n}{\log |z_n|}$

$\Rightarrow \sigma \geq \limsup\limits_{n \to \infty} \dfrac{\log n}{\log |z_n|}$ ——— (1) $\{\because \alpha > \sigma\}$

Next; let α' be an arbitrary number exceeding the RHS of (1).

Then there exists $n > N$ such that

$$\frac{\log n}{\log |z_n|} < \alpha', \quad \forall\ n > N$$

$\Rightarrow \log n < \alpha' \log |z_n|$

$\Rightarrow \log n < \log |z_n| \alpha'$

$\Rightarrow n < |z_n| \alpha'$

$\Rightarrow n^{1/\alpha'} < |z_n|$

$\Rightarrow n^{-1/\alpha'} > |z_n|^{-1} \quad \forall \; n > N$

This means that the series $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{\beta}}$ converges for every $\beta > \alpha'$

It follows from the exponent convergence that $\sigma \leq \alpha'$.

Hence by the definition of α', we have

$$\sigma \leq \limsup_{n\to\infty} \frac{\log n}{\log |z_n|} \quad \text{—— (2)}$$

Comparing (1) and (2), we get

$$\sigma = \limsup_{n\to\infty} \frac{\log n}{\log |z_n|}$$

This completes the proof.

Theorem:- If $f(z)$ is an entire function of order ρ & convergence exponent σ, then

$$\sigma \leq \rho$$

Proof:- Suppose ρ is infinite

Then the inequality $\sigma \leq \rho$ is trivial.

Again if the number of zeros is finite then $\sigma = 0$
and so $\sigma \leq \rho$ holds.

So we may suppose that ρ is finite and that there are infinitely many zeros which we arrange as a sequence $\{z_n\}$ such that
$|z_1| \leq |z_2| \leq \cdots \leq |z_n| \leq |z_{n+1}| \leq \cdots$
and $|z_n| \to \infty$ as $n \to \infty$

Let $n(r)$ denote the number of zeros of an entire function $f(z)$ in the closed disc $|z| \leq r$
Then $n(|z_n|) \geq n$ —— (1)

The strict inequality $n(|z_n|) > n$ will hold if $|z_n| = |z_{n+1}|$

Also note that $|z_n| \to \infty$ as $n \to \infty$
therefore we have,
$$n(|z_n|) \leq |z_n|^{\rho+\epsilon} \qquad \text{—— (2)}$$
where $\epsilon > 0$, and n is sufficiently large

Then (1) & (2) gives
$$|z_n|^{\rho+\epsilon} \geq n, \quad \text{for } n \text{ sufficiently large} \qquad \text{—— (3)}$$

Since $|z_n| \to \infty$, we may assume that $|z_n| \geq 1$

For given $\alpha > \rho$, we may take $\epsilon < \alpha - \rho$, so that
$\rho + \epsilon < \alpha$
$$\Rightarrow 1 < \frac{\alpha}{\rho+\epsilon} \quad \text{or} \quad \frac{\alpha}{\rho+\epsilon} > 1$$

Then for n sufficiently large, we have from (3)

$$|z_n| \geq n^{1/\rho+\epsilon}$$

$$\Rightarrow |z_n|^{\alpha} \geq n^{\alpha/\rho+\epsilon}$$

$$\Rightarrow |z_n|^{-\alpha} \leq n^{-\alpha/\rho+\epsilon}$$

$$\Rightarrow \sum_{n=1}^{\infty} |z_n|^{-\alpha} \leq \sum_{n=1}^{\infty} n^{-\alpha/\rho+\epsilon} \quad \text{---} (4)$$

Since $\frac{\alpha}{\rho+\epsilon} > 1$, the series $\sum_{n=1}^{\infty} n^{-\alpha/\rho+\epsilon}$ is convergent

and so $\sum_{n=1}^{\infty} n^{-\alpha/\rho+\epsilon} < \infty$

We thus conclude from 4 that

$$\sum_{n=1}^{\infty} |z_n|^{-\alpha} < \infty \quad \forall \ \alpha > \rho \quad \text{---} (5)$$

Since by definition of exponent of convergence, we have

$$\sigma = \inf \left\{ \alpha > 0 : \sum_{n=1}^{\infty} |z_n|^{-\alpha} < \infty \right\}$$

and since (5) holds for every $\alpha > \rho$

We conclude that,

$$\boxed{\sigma \leq \rho}$$

This completes the proof of theorem.

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Theorem:- (a) If $f \in \mathcal{S}$, $|\alpha| = 1$ and ~~ane~~ $g(z) = \bar{\alpha} f(\alpha, z)$, then $g \in \mathcal{S}$

(b) If $f \in \mathcal{S}$ there exists a $g \in \mathcal{S}$ such that

$$g^2(z) = f(z^2) \quad (z \in U) \qquad \text{—— (1)}$$

Proof:- (a) It is given that

$$f \in \mathcal{S}, \quad |\alpha| = 1 \quad \text{and} \quad g(z) = \bar{\alpha} f(\alpha, z)$$

To prove that g is one to one

$$\begin{aligned}
g(z_1) = g(z_2) &\Rightarrow \bar{\alpha} f(\alpha z_1) = \bar{\alpha} f(\alpha z_2) \\
&\Rightarrow \bar{\alpha} [f(\alpha z_1) - f(\alpha z_2)] = 0 \\
&\Rightarrow f(\alpha z_1) - f(\alpha z_2) = 0 \qquad \{\because |\alpha| \neq 0\} \\
&\Rightarrow \alpha z_1 = \alpha z_2 \\
&\Rightarrow \alpha (z_1 - z_2) = 0 \\
&\Rightarrow z_1 - z_2 = 0 \qquad \{\because |\alpha| \neq 0\} \\
&\Rightarrow z_1 = z_2
\end{aligned}$$

Hence g is one to one.

Also, $\because g(z) = \bar{\alpha} f(\alpha z)$

$$\begin{aligned}
\therefore g(0) &= \bar{\alpha} f(0) \\
\Rightarrow g(0) &= 0 \qquad (\because f(0) = 0)
\end{aligned}$$

and $\because g(z) = \bar{\alpha} f(\alpha z)$

$$\begin{aligned}
\therefore g'(z) &= \bar{\alpha} \alpha f'(\alpha z) \\
&= |\alpha|^2 f'(\alpha z) \\
\Rightarrow g'(z) &= f'(\alpha z) \qquad \{\because |\alpha| = 1\} \\
\Rightarrow g'(0) &= f'(0) \\
\Rightarrow g'(0) &= 1 \qquad \{\because f'(0) = 1\}
\end{aligned}$$

Hence $g \in \mathcal{S}$

(b) It is given that $f \in \mathcal{J}$

$\because f \in \mathcal{J}$

$\therefore f \in H(U)$

f is one to one, $f(0) = 0$ and $f'(0) = 1$

Define $\varphi(z) = \frac{f(z)}{z}$ —— (1)

Clearly, $f(z)$ and z both have simple zero at 0 then $\frac{f(z)}{z}$ has a removable singularity at 0, by suitably defining $\varphi(z)$ at 0 so we define $\varphi(0) = 1$

Then $\frac{f(z)}{z}$ is holomorphic at 0

Thus $\varphi \in H(U)$ and $\varphi(0) = 1$

Now, since f has no zero in $U - \{0\}$, therefore φ has no zero in U

Hence $\exists$ an $h \in H(U)$ with $h(0) = 1$ & $h^2(z) = \varphi(z)$

Now, put $g(z) = z\, h(z^2)$ $(z \in U)$ —— ②

Then $g^2(z) = z^2 h^2(z^2)$

$= z^2 \varphi(z^2)$

$\Rightarrow g^2(z) = f(z^2)$ (from ①)

Now, to prove that $g \in \mathcal{J}$

Now, since $g(z) = \cancel{z(h^2)}\ z\, h(z^2)$

$\therefore \quad g(0) = 0\, h(0)$

$\Rightarrow \quad g(0) = 0$

and $\quad g^2(z) = f(z^2)$

$\Rightarrow \quad g(z) = \sqrt{f(z^2)}$

$\Rightarrow \quad g'(z) = \frac{1}{2} f(z^2)^{-1/2} f'(z^2).\, 2z$

$$= \frac{2z\, f'(z^2)}{2\sqrt{f(z^2)}}$$

$$= \frac{z\, f'(z^2)}{\sqrt{f(z^2)}}$$

$$= \frac{z\, f'(z^2)}{\sqrt{z^2 + \sum_{n=2}^{\infty} a_n z^{2n}}}$$

$$\Rightarrow g'(z) = \frac{f'(z^2)}{\sqrt{1 + \sum_{n=2}^{\infty} a_n z^{2n-2}}}$$

$$\Rightarrow g'(0) = \frac{f'(0)}{\sqrt{1+0}}$$

$\Rightarrow \quad g'(0) = 1 \qquad \{\because f'(0) = 1\}$

Finally we shall show that g is one-to-one

Suppose $z_1, z_2 \in U$ then

$g(z_1) = g(z_2) \Rightarrow g^2(z_1) = g^2(z_2)$

$\Rightarrow f(z_1^2) = f(z_2^2)$

$\Rightarrow z_1^2 = z_2^2 \qquad \{\because g^2(z) = f(z^2)\}$

$\Rightarrow z_1 = \pm z_2$

$\Rightarrow z_1 = z_2 \quad$ or $\quad z_1 = -z_2$

If $z_1 = -z_2$, then from (2),

$$g(z_1) = -g(z_2)$$

It follows that $g(z_1) = g(z_2) = 0$ and since g has no zero in $U - \{0\}$

$\therefore z_1 = z_2 = 0$

i.e. g is one-one

Hence $g \in \varphi$

The Great Picard theorem

Statement:- Let f be an analytic function that has an essential singularity at $z = z_0$. Then in each neighbourhood of z_0, f assumes each complex numbers, with one possible exception, an infinite number of times.

Proof:- Let f be an analytic function that has an essential singularity at $z = z_0$

So we may assume that f has an essential singularity at $z = 0$.

Now suppose if possible, there is an $R > 0$ such that there are two numbers not in $\{f(z) : 0 < |z| < R\}$

Let $f(z) \neq a$ and $f(z) \neq b$ for all $z \in B(0, R)$ and define $F(z) = \dfrac{f(z) - a}{b - a}$

Then $F(z)$ omits the values 0 and 1, i.e. $F(z) \neq 0$ and $F(z) \neq 1$ for all z.

So we may suppose that $f(z) \neq 0$ and $f(z) \neq 1$ for $0 < |z| < R$

Let $G = B(0, R) - \{0\}$ and define $f_n : G \to \mathbb{C}$ by

$$f_n(z) = f\left(\frac{z}{n}\right)$$

Since f is analytic and do not assume the values 0 and 1, therefore f_n also analytic in G and do not assume the values 0 and 1

Now, by Montel-Caratheodary theorem, $\{f_n\}$ is a normal family in $C(G, \mathbb{C}_\infty)$

Thus for any sequence $\{f_n\}_{n=1}^{\infty}$ there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \to \varphi$ uniformly on $B(0; \frac{1}{2}R)$ where φ is either analytic in G or $\varphi \equiv \infty$

If φ is analytic,
Let $M = \max\{|\varphi(z)| : |z| = \frac{1}{2}R\}$
then for any $\epsilon = M > 0$, $\exists\ u \in \mathbb{N}$ such that for all $k \geq u$, we have

$$\left|f\left(\frac{z}{n_k}\right)\right| = |f_{n_k}(z)|$$

$$= |f_{n_k}(z) - \varphi(z) + \varphi(z)|$$

$$\leq |f_{n_k}(z) - \varphi(z)| + |\varphi(z)|$$

$$\leq \sup_{z \in B(0; \frac{1}{2}R)} |f_{n_k}(z) - \varphi(z)| + \max_{|z| = \frac{1}{2}R} |\varphi(z)|$$

$$\leq \epsilon + M \qquad \left\{\begin{array}{l} \because f_{n_k} \to \varphi \\ \therefore |f_{n_k}(z) - \varphi(z)| < \epsilon \end{array}\right\}$$

$$= M + M$$

$$= 2M$$

Thus $\left|f\left(\frac{z}{n_k}\right)\right| \leq 2M$

Hence $|f(z)| \leq 2M$ for sufficiently large n_k and $|z| = \frac{R}{2n_k}$

Now, by Maximum modulus principle, f is uniformly bounded on concentric annuli about zero

$\Rightarrow$ f is bounded by 2M on a deleted nbd of zero
$\Rightarrow$ $z=0$ must be a removable singularity

Therefore, φ cannot be analytic and so $\varphi \equiv \infty$

Also, if $\varphi \equiv \infty$, then f must ~~be~~ have a pole at zero.

Hence, we conclude that there is almost one complex number that is never assumed by f

Hadmard's Factorization Theorem

Statement:- If f is an entire function of finite order ρ then f has finite genus $\mu \leq \rho$

Proof:- By Weierstrass factorization theorem, the entire function f(z) can be represented in the form given by,

$$f(z) = z^m e^{g(z)} p(z) \quad \text{——— (1)}$$

where g(z) is an entire function.

It is evident that the division of f(z) by cz^m does not affect either the hypothesis or the conclusion of the theorem and so it is sufficient to consider the representation

$$f(z) = e^{g(z)} p(z)$$

$$\Rightarrow e^{g(z)} = \frac{f(z)}{p(z)}$$

$$\Rightarrow |e^{g(z)}| = \left|\frac{f(z)}{p(z)}\right|$$

$$\Rightarrow e^{Re\, z} = \frac{|f(z)|}{|p(z)|}$$

$$\Rightarrow \log e^{Re\, z} = \log|f(z)| - \log|p(z)|$$

$$\Rightarrow Re\, z = \log|f(z)| - \log|p(z)| \quad \text{—— (2)}$$

By the definition of order, we have

$$\rho = \inf\{\lambda : M(r) \leq \exp(r^{\lambda}) \text{ for sufficiently large } r\}$$

$$\Rightarrow \rho + \epsilon > \lambda, \text{ with } M(r) \leq \exp(r^{\lambda}) \quad (\text{for any } \epsilon > 0)$$

$$\Rightarrow \lambda < \rho + \epsilon \quad \text{with } M(r) \leq \exp(r^{\lambda})$$

$$\Rightarrow M(r) \leq \exp(r^{\rho+\epsilon})$$

$$\Rightarrow \max\{|f(z)|\} < \exp(r^{\rho+\epsilon})$$

$$\{\because M(r) = \max\{|f(z)| : |z| = r\}\}$$

$$\Rightarrow |f(z)| \leq \exp(r^{\rho+\epsilon})$$

$$\Rightarrow \log|f(z)| \leq r^{\rho+\epsilon} \quad \text{—— (3)}$$

Suppose σ is the convergence exponent of non-zero zeros of $f(z)$ then $\sigma \leq \rho$

Also, by Borel's theorem σ is the order of the

canonical product $p(z)$,
and so,
$$\log|p(z)| > -r^{\sigma+\epsilon} \quad \text{for large } |z|=r$$
$$\Rightarrow -\log|p(z)| < r^{\sigma+\epsilon} \quad \text{--- (4)}$$

Adding (3) and (4),
$$\log|f(z)| - \log|p(z)| < r^{\rho+\epsilon} + r^{\sigma+\epsilon} \quad \text{--- (5)}$$

Combining (2) and (5), we get
$$\operatorname{Re} g(z) < r^{\rho+\epsilon} + r^{\sigma+\epsilon}$$
$$\Rightarrow \operatorname{Re} g(z) < r^{\rho+\epsilon} + r^{\rho+\epsilon} \quad (\because \sigma \le \rho)$$
$$\Rightarrow \operatorname{Re} g(z) < 2r^{\rho+\epsilon} \quad \text{--- (6)}$$

Since r is large, we conclude that $g(z)$ is a polynomial of degree not exceeding ρ.

So the genus of $f \le \rho$, i.e. $\boxed{\mu \le \rho}$

Proved.

Example:- Use Hadmard's factorization theorem to show that $\sin \pi z = \pi z \sum_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$

Hence deduce Walli's product formula
$$\frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8 \cdots 2n.2n}{1.3.3.5.5.7.7.9 \cdots (2n-1)(2n+1)}$$

Solution:- The zeros of $\sin \pi z$ are given by
$$\sin \pi z = 0 = \sin n\pi$$
$\therefore z = 0, \pm 1, \pm 2, \pm 3, \ldots$ i.e. $\{z_n\} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$
$= \{\pm n\}_{n=-\infty}^{\infty}$

Thus the genus of the canonical product is clearly 1. since $h=1$ is the smallest integer such that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{h+1}} = \sum_{n=1}^{\infty} \frac{1}{|n|^{1+1}} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

Hence, the canonical product associated with the non-zero zeros of $\sin \pi z$ is of the form

$$p(z) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \quad (n \neq 0)$$

Combining the factors corresponding to n and $-n$, we get

$$p(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Further, the order of $\sin \pi z$ is 1.

Since $z=0$ is a simple zero of $\sin \pi z$, therefore Hadmard's factorization of $\sin \pi z$ may be written as

$$\sin \pi z = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

where $g(z)$ is a polynomial of degree atmost 1 (the order of $\sin \pi z$)

Thus we may write $g(z) = a_0 + a_1 z$

Hence $\sin \pi z = z e^{a_0 + a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ —— ①

To determine a_0 and a_1 :-

We write $\dfrac{\sin \pi z}{z} = e^{a_0 + a_1 z} \cdot \prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2}\right)$

$\Rightarrow \pi \cdot \dfrac{\sin \pi z}{\pi z} = e^{a_0 + a_1 z} \prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2}\right)$

$\Rightarrow \pi \lim_{z \to 0} \dfrac{\sin \pi z}{\pi z} = \lim_{z \to 0} \left[e^{a_0 + a_1 z} \prod_{n=1}^{\infty} \left(1 - \dfrac{z^2}{n^2}\right)\right]$

$\Rightarrow \pi \cdot 1 = e^{a_0} \cdot 1$

$\Rightarrow \boxed{e^{a_0} = \pi}$

Therefore by ①,

$$\sin \pi z = z \pi e^{a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\Rightarrow \frac{\sin \pi z}{z} = \pi e^{a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad \text{——— ②}$$

Replacing z by $-z$ in ②, we get

$$\frac{\sin \pi(-z)}{(-z)} = \pi e^{-a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\Rightarrow \frac{\sin \pi z}{z} = \pi e^{-a_1 z} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad \text{——— ③}$$

Comparing (2) and (3), we obtain

$e^{a_1 z} = e^{-a_1 z} \Rightarrow e^{2a_1 z} = 1 \quad \forall z$

$\Rightarrow \log e^{2a_1 z} = \log 1$

$\Rightarrow 2a_1 z = 0$

$\Rightarrow \boxed{a_1 = 0}$

Hence $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ —— (4)

Deduction:- Put $z = \frac{1}{2}$ in (4), we obtain

$$\sin \frac{\pi}{2} = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{(1/2)^2}{n^2}\right)$$

$$\Rightarrow 1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(\frac{4n^2-1}{4n^2}\right)$$

$$\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{4n^2-1}\right)$$

$$\Rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n.2n}{(2n-1)(2n+1)}$$

$$\Rightarrow \frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8 \cdots 2n.2n}{1.3.3.5.5.7.7.9 \cdots (2n-1)(2n+1)}$$

Proved.

Unit-5

5'' **Lemma 1:-** Let f be analytic in $\mathcal{D} = \{z : |z| < 1\}$ and let $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M$ for all z in $\mathcal{D}$. Then $M \geq 1$ and

$$f(\mathcal{D}) \supset B\left(0 ; \frac{1}{6M}\right)$$

Proof:- Given:- f is analytic in $\mathcal{D} = \{z : |z| < 1\}$ and $f(0) = 0$, $f'(0) = 1$, $|f(z)| \leq M \ \forall z \in D$

Let $0 < r < 1$.

Since f is analytic in $\mathcal{D}$, therefore by Maclaurin series of $f(z)$ is given by

$$f(z) = f(0) + f'(0) z + \sum_{n=2}^{\infty} \frac{f^{(n)}(0)}{\lfloor n} z^n \quad \text{—— (1)}$$

$$\Rightarrow f(z) = 0 + z + \sum_{n=2}^{\infty} \frac{f^{(n)}(0)}{\lfloor n} z^n$$

$$\Rightarrow f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \left(\text{Take } \frac{f^{(n)}(0)}{\lfloor n} = a_n\right)$$

Then By Cauchy's estimate we have

$$|a_n| = \frac{|f(z)|}{|z|^n} \leq \frac{M}{r^n}, \quad \text{for } n \geq 1$$

So,

$$|a_1| \leq \frac{M}{r}$$

$$\Rightarrow 1 \leq M$$

$$\Rightarrow \boxed{M \geq 1}$$

If $|z| = \frac{1}{4M}$

Then $|f(z)| = \left|z + \sum_{n=2}^{\infty} a_n z^n\right|$

$$\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n$$

$$\geq \frac{1}{4M} - \sum_{n=2}^{\infty} \frac{M}{r^n} \cdot \frac{1}{(4M)^n}$$

$$\geq \frac{1}{4M} - \sum_{n=2}^{\infty} \frac{1}{4^n M^{n-1}} \qquad (\because r < 1)$$

$$= \frac{1}{4M} - \frac{1/16M}{1 - 1/4M}$$

$$= \frac{1}{4M} - \frac{1}{16M - 4}$$

$$= \frac{1}{4M} - \frac{1}{16M - 4M} \qquad (\because M \geq 1)$$

$$= \frac{1}{4M} - \frac{1}{12M}$$

$$= \frac{2}{12M} = \frac{1}{6M}$$

i.e. $\boxed{|f(z)| \geq \frac{1}{6M}}$

Suppose $|w| < \frac{1}{6M}$ and set $g(z) = f(z) - w$

Now we will show that $g(z)$ has a zero;

We have,

$$|f(z) - g(z)| = |w|$$

$$< \frac{1}{6M}$$

$$\leq |f(z)|$$

$$\Longrightarrow |f(z) - g(z)| < |f(z)|$$

Hence, by Rouche's Theorem, f and $f+(g-f)=g$ have the same number of zeros in $B\left(0;\frac{1}{4M}\right)$

Again, since $f(0)=0$,

$\therefore\ g(z_0)=0$, for some $z_0 \in B\left(0;\frac{1}{4M}\right)$

But $g(z_0)=0$ yields $f(z_0)=w$
and $|f(z_0)| = |w| < \frac{1}{6M}$

Hence we conclude that

$$\boxed{f(D) \supset B\left(0,\frac{1}{6M}\right)}$$

Proved

Lemma 2:- Let g be analytic in $B(0;R)$, $g(0)=0$, $|g'(0)| = \mu > 0$ and $|g(z)| \le M$ for all z then $g(B(0;R)) \supset B\left(0;\frac{R^2\mu^2}{6M}\right)$

Proof:- Let $f(z) = [R g'(0)]^{-1} g(Rz)$ for $|z|<1$.

Since g is analytic in $B(0;R)$;
$\therefore\ g(z)$ is analytic in $B(0;R) = \{z : |z|<R\}$

$\Rightarrow g(Rz)$ is analytic in $\mathcal{D} = \{z : |z|<1\}$
$\Rightarrow [R g'(0)]^{-1} g(Rz)$ is analytic in $\mathcal{D} = \{z : |z|<1\}$
$\Rightarrow f(z)$ is analytic in $\mathcal{D} = \{z : |z|<1\}$

Further, $g(0)=0 \Rightarrow f(0)=0$

Also, $f'(z) = [R\, g'(0)]^{-1} g'(Rz) . R$
$\Rightarrow f'(z) = [g'(0)]^{-1} g'(Rz)$

so that, $f'(0) = [g'(0)]^{-1} g'(0)$
$= 1$ $\{\because |g'(0)| \neq 0\}$

Furthermore,
$$|f(z)| = \frac{|g(Rz)|}{R|g'(0)|} \leq \frac{M}{\mu R} \text{ for all } z \text{ in } D$$

Then we have,
$$f(D) \supset B\left(0, \frac{\mu R}{6M}\right) \text{ --- } (1)$$

Now,
$$B\left(0; \frac{R^2\mu^2}{6M}\right) = \left\{z : |z| < \frac{R^2\mu^2}{6M}\right\}$$
$$= \left\{z : \left|\frac{z}{R\mu}\right| < \frac{\mu R}{6M}\right\}$$
$$= \left\{R\mu z : |z| < \frac{\mu R}{6M}\right\}$$
$$= R\mu\left\{z : |z| < \frac{\mu R}{6M}\right\} \subset R\mu f(D) \text{ --- } (2)$$
$$= R\mu\{f(z) : |z| < 1\}$$
$$= \{R g'(0) f(z) : |z| < 1\}$$
$$= \{g(Rz) : |z| < 1\}$$
$$= \left\{g(z) : \left|\frac{z}{R}\right| < 1\right\}$$
$$= \{g(z) : |z| < R\}$$
$$= g(B(0; R))$$
$$\Rightarrow g(B(0;R)) \supset B\left(0; \frac{R^2\mu^2}{6M}\right).$$

Proved

Bloch's Theorem:-

Statement:- Let f be an analytic function in a region containing the closure of the disc $\mathcal{D} = \{z : |z| < 1\}$ and satisfying $f(0) = 0$, $f'(0) = 1$. Then there is a disc $S \subset \mathcal{D}$ in which f is one-one and such that $f(S)$ contains a disc of radius $1/72$.

Proof:- Let $k(r) = \max\{|f'(z)| : |z| = r\}$ ——— ①
and ~~h(r~~ take $h(r) = (1-r)\,k(r)$ ——— ②

Then $h : [0, 1] \to R$ is continuous
and $h(0) = k(0) = \max\{|f'(0)|\}$
$= \max\{1\}$
$= 1$

and $h(1) = 0$

Let $r_0 = \sup\{r : h(r) = 1\}$
then $r \le r_0$, $h(r_0) = 1$, $r_0 < 1$;
for if $r_0 = 1$ then $h(r_0) = 0$.

Now since $\underset{0 \le r \le 1}{\text{Max}}\, h(r) = 1$ and for all $r \le r_0$, $h(r) = 1$

$\therefore$ $h(r) < 1$, if $r > r_0$

Let z_0 be so choosen that $|z_0| = r_0$ and $|f'(z_0)| = k(r_0)$, then

$$h(r_0) = (1-r_0)\,k(r_0)$$
$$\Rightarrow 1 = (1-r_0)\,k(r_0)$$
$$\Rightarrow k(r_0) = (1-r_0)^{-1}$$
$$\Rightarrow |f'(z_0)| = (1-r_0)^{-1} \quad \text{——— ③}$$

Now if $|z-z_0| < \frac{1}{2}(1-r_0) = \rho_0$

Then $\rho_0 > |z-z_0|$

$\Rightarrow \rho_0 > |z| - |z_0|$

$\Rightarrow \rho_0 + |z_0| > |z|$

$\Rightarrow \frac{1}{2}(1-r_0) + |z_0| > |z|$

$\Rightarrow \frac{1}{2}(1-r_0) + r_0 > |z|$ $\{\because |z_0| = r_0\}$

$\Rightarrow \frac{1}{2} + \frac{1}{2}r_0 > |z|$

$\Rightarrow |z| < \frac{1}{2}(1+r_0)$

$\Rightarrow r < \frac{1}{2}(1+r_0)$ $\{\because |z| = r\}$

Also, $|f'(z)| \leq k(r)$

$< k\left(\frac{1}{2}(1+r_0)\right)$

$= \left[1 - \frac{1}{2}(1+r_0)\right]^{-1} h\left(\frac{1}{2}(1+r_0)\right)$ $\{$by 2$\}$

$< \left[1 - \frac{1}{2}(1+r_0)\right]^{-1}$ $\{\because h(r) < 1$ if $r > r_0\}$

$= \left[\frac{1}{2}(1-r_0)\right]^{-1}$

$= \rho_0^{-1}$

$\Rightarrow |f'(z)| < \frac{1}{\rho_0}$ —— 4

Thus we get,

$|f'(z) - f'(z_0)| \leq |f'(z)| + |f'(z_0)|$

$$< \frac{1}{\rho_0} + \frac{1}{1-r_0} \quad \{\text{from } 4 \text{ and } 3\}$$

$$= \frac{1}{\rho_0} + \frac{1}{2\rho_0} \quad \left\{\because \frac{1}{2}(1-r_0) = \rho_0\right\}$$

$$= \frac{3}{2\rho_0}$$

$$\Rightarrow |f'(z) - f'(z_0)| < \frac{3}{2\rho_0} \quad \text{——} \quad (5)$$

Let us consider the transformation $Z = \dfrac{z-z_0}{\rho_0}$

and define $F(Z) = \dfrac{f'(z) - f'(z_0)}{3/2\rho_0}$

Then $|Z| = \left|\dfrac{z-z_0}{\rho_0}\right| = \dfrac{|z-z_0|}{\rho_0} < \dfrac{\rho_0}{\rho_0} = 1 \quad \{\because |z-z_0| < \rho_0\}$

and $|F(Z)| = \left|\dfrac{f'(z) - f'(z_0)}{3/2\rho_0}\right|$

$$= \frac{|f'(z) - f'(z_0)|}{3/2\rho_0}$$

$$< \frac{3/2\rho_0}{3/2\rho_0} = 1 \quad \{\text{from } (5)\}$$

Thus we have $|Z| < 1$ and $|F(Z)| < 1$
and $F(0) = 0$

Hence by Schwarz's Lemma,
$|F(Z)| < |Z|$, for Z in $\mathcal{D} = \{Z : |Z| < 1\}$

$$\Rightarrow \left|\frac{f'(z) - f'(z_0)}{3/2\rho_0}\right| < \left|\frac{z-z_0}{\rho_0}\right| \quad \text{for } z \in B(z_0, \rho_0)$$

$$\Rightarrow |f'(z) - f'(z_0)| < \frac{3}{2\rho_0} \frac{|z-z_0|}{\rho_0} = \frac{3|z-z_0|}{2\rho_0^2}$$

for $z \in B(z_0;\rho_0)$

Let $S = B(z_0; \frac{1}{3}\rho_0)$

so if $z \in S$ then $|z-z_0| < \frac{1}{3}\rho_0$

And so from (6), we get

$$|f'(z) - f'(z_0)| < \frac{3}{2\rho_0^2}\left(\frac{1}{3}\rho_0\right) = \frac{1}{2\rho_0}$$

$$\Rightarrow |f'(z) - f'(z_0)| < \frac{1}{2\rho_0} = |f'(z_0)| \quad \text{(from (3))}$$

(7)

$$\Rightarrow |f'(z) - f'(z_0)| < |f'(z_0)|$$

And hence f is one-one in S (By lemma (3))

Now, it remains to show that $f(S)$ contains a disc of radius $\frac{1}{72}$.

So let us define $g: B\left(0; \frac{1}{3}\rho_0\right) \to \mathbb{C}$ by
$g(z) = f(z+z_0) - f(z_0)$

Then $g(0) = 0$
and $g'(z) = f'(z+z_0)$
$\therefore |g'(0)| = |f'(z_0)|$
$\Rightarrow |g'(0)| = \frac{1}{2\rho_0}$

Also, if $z \in B\left(0; \frac{1}{3}\rho_0\right)$, then the line segment

$\gamma = [z_0, z+z_0]$ lies in $S \subset B(z_0; \rho_0)$

Thus, $g(z) = f(z+z_0) - f(z_0)$

$$\Rightarrow |g(z)| = |f(z+z_0) - f(z_0)|$$

$$\Rightarrow |g(z)| = \left| \int_\gamma f'(w)\, dw \right|$$

$$\Rightarrow |g(z)| \leq \int_\gamma |f'(w)|\, |dw| \qquad \{\text{from } 4\}$$

$$\Rightarrow |g(z)| < \int_\gamma \frac{1}{\rho_0} |dw| \qquad \left\{ \because |f'(z)| < \frac{1}{\rho_0} \text{ whenever } |z-z_0| < \rho_0 \right\}$$

$$\Rightarrow |g(z)| < \frac{1}{\rho_0} \int_\gamma |dw|$$

$$\Rightarrow |g(z)| < \frac{1}{\rho_0} \left| [(z+z_0) - z_0] \right|$$

$$\Rightarrow |g(z)| < \frac{1}{\rho_0} |z|$$

$$\Rightarrow |g(z)| < \frac{1}{\rho_0} \cdot \frac{1}{3}\rho_0 = \frac{1}{3} \qquad \left\{ \begin{array}{l} \because z \in B(0; \frac{1}{3}\rho_0) \\ \therefore |z| = \frac{1}{3}\rho_0 \end{array} \right\}$$

$$\Rightarrow |g(z)| < \frac{1}{3} \quad \text{for all } z \in B(0; \tfrac{1}{3}\rho_0)$$

And hence $g\left(B(0; \frac{1}{3}\rho_0)\right) \supset B(0; \sigma)$ (By lemma 2)

where $\sigma = \dfrac{\left(\frac{1}{3}\rho_0\right)^2 \left(\frac{1}{2\rho_0}\right)^2}{6(1/3)} = \dfrac{1}{72}$

Now, $B\left(f(z_0); \frac{1}{72}\right) = \left\{ w : |w - f(z_0)| < \frac{1}{72} \right\}$

$$= \left\{ f(z+z_0) : |f(z+z_0) - f(z_0)| < \frac{1}{72} \right\}$$

$$= \left\{ g(z) + f(z_0) : |g(z) - 0| < \frac{1}{72} \right\}$$

$$= \left\{ g(z) : |g(z) - 0| < \frac{1}{72} \right\} + f(z_0)$$

$$= \left\{ \omega : |\omega - 0| < \frac{1}{72} \right\} + f(z_0)$$

$$= \left\{ \omega : |\omega| < \sigma \right\} + f(z_0) \qquad \left\{ \because \sigma = \frac{1}{72} \right\}$$

$$= B(0;\sigma) + f(z_0)$$

$$\subset g\left(B\left(0; \tfrac{1}{3}\rho_0\right)\right) + f(z_0)$$

$$= \left\{ g(z) : |z| < \frac{1}{3}\rho_0 \right\} + f(z_0)$$

$$= \left\{ g(z) : |z + z_0 - z_0| < \frac{1}{3}\rho_0 \right\} + f(z_0)$$

$$= \left\{ g(z - z_0) : |z - z_0| < \frac{1}{3}\rho_0 \right\} + f(z_0)$$

$$= \left\{ f(z) - f(z_0) : |z - z_0| < \frac{1}{3}\rho_0 \right\} + f(z_0)$$

$$= \left\{ f(z) : |z - z_0| < \frac{1}{3}\rho_0 \right\}$$

$$= f\left(B\left(z_0; \tfrac{1}{3}\rho_0\right)\right)$$

$$= f(S)$$

i.e., $f(S) \supset B\left(f(z_0); \frac{1}{72}\right)$

This completes the proof of the theorem.

Borel's Theorem

Statement:- The order of a canonical product is equal to the convergence exponent of its zeros.

Proof: Let ρ and σ denote the order and convergence exponent of a canonical product $p(z)$.

In order to prove $\rho = \sigma$, first we shall show that $\sigma \leq \rho$ (prove by previous theorem)

Now, to prove the reverse inequality, consider the Weierstrass primary factors,

$$E(w, p) = (1-w)\exp\left\{w + \frac{1}{2}w^2 + \cdots + \frac{1}{p}w^p\right\}$$

If $p > 0$, then $\exists$ $a, b > 0$ such that

$$|E(w,p)| \leq e^b \exp(a|w|^p)$$

$$\Rightarrow |E(w,p)| \leq \exp(b + a|w|^p)$$

Thus if $c = a+b$, $|w| \geq 1$, $\alpha \geq p$

Then we have,

$$\begin{aligned} |E(w,p)| &\leq \exp\{b.1 + a|w|^p\} \\ &\leq \exp\{(b+a)|w|^p\} \\ &= \exp\{c|w|^p\} \\ &\leq \exp\{c|w|^\alpha\} \qquad (\alpha \geq p) \text{ --- } (1) \end{aligned}$$

Also, if $|w| \leq \frac{1}{2}$ and $p \geq 0$, then

$$|E(w,p)| \leq \exp(2|w|^{p+1})$$

$$\Rightarrow |E(w,p)| \leq \exp(2|w|^\alpha), \text{ for } \alpha \leq p+1, |w| \leq \frac{1}{2} \text{ --- } (2)$$

Again, if $\frac{1}{2} \leq |w| \leq 1$ and $p \geq 0$, then

$$|E(w,p)| \leq \exp\left(c|w|^{p+1}\right)$$

Since $|w| \leq 1$

$\therefore |E(w,p)| \leq \exp\left(c|w|^{\alpha}\right)$, for $\alpha \leq p+1$, $\frac{1}{2} \leq |w| \leq 1$ —— (3)

Now from (1), (2) and (3) we conclude that for $p \leq \alpha \leq p+1$ then, $\exists$ c such that

$$|E(w,p)| \leq \exp\left(c|w|^{\alpha}\right) \quad \text{—— (4)}$$

Let $\{z_n\}_{n=1}^{\infty}$ be the sequence of zeros of the canonical product

$$p(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, h\right)$$

Then h is the genus of the canonical product and satisfies $h \leq \sigma \leq h+1$

If $\sigma = h+1$, then we assume that $\alpha = h+1$, while if $\sigma < h+1$, we let $\sigma < \alpha \leq h+1$

So, by the definition of exponent of convergence we have,

$$\sigma = \inf\left\{\lambda > 0 : \sum_{n=1}^{\infty} |z_n|^{-\lambda} < \infty\right\}$$

Thus, $\sum_{n=1}^{\infty} |z_n|^{-\alpha} < \infty$

Let $\sum_{n=1}^{\infty} |z_n|^{-\alpha} = k$

Hence, if $p_m(z) = \prod_{n=1}^{m} E\left(\frac{z}{z_n}, h\right)$

Then,
$$|p_m(z)| = \left|\prod_{n=1}^{m} E\left(\frac{z}{z_n}, h\right)\right|$$

$$\Rightarrow |p_m(z)| = \prod_{n=1}^{m} \left|E\left(\frac{z}{z_n}, h\right)\right|$$

$$\Rightarrow |p_m(z)| \leq \prod_{n=1}^{m} \exp\left(c\left|\frac{z}{z_n}\right|^{\alpha}\right) \quad (\text{by eq}^n \text{ 4})$$

$$\Rightarrow \log|p_m(z)| \leq \sum_{n=1}^{m} \log \exp\left(c\left|\frac{z}{z_n}\right|^{\alpha}\right)$$

$$\Rightarrow \log|p_m(z)| \leq \sum_{n=1}^{m} \left(c\left|\frac{z}{z_n}\right|^{\alpha}\right)$$

$$\Rightarrow \log|p_m(z)| \leq c|z|^{\alpha} \sum_{n=1}^{\infty} |z_n|^{-\alpha}$$

$$\Rightarrow \log|p_m(z)| \leq c|z|^{\alpha} k$$

$$\Rightarrow |p_m(z)| \leq \exp\left(c|z|^{\alpha} k\right) \quad \text{--- } (5)$$

$\forall\ \alpha > \sigma$ and $\forall z$

Hence we conclude from (5) that
$$\rho \leq \sigma$$

Hence $\boxed{\rho = \sigma}$

This completes the proof of the theorem.

$H(G)$ - collection of analytic functions in G

Montel Caratheodory Theorem :-

<u>Statement</u>:- Let $\mathcal{F}$ be the family of all analytic functions defined in a region G that do not assume the values 0 and 1 then $\mathcal{F}$ is normal in $C(G, \mathbb{C}_\infty)$.

<u>Proof</u>:- Choose a point z_0 in G and keep it fix.

Define $\mathcal{G} = \{ f \in \mathcal{F} : |f(z_0)| \leq 1 \}$

and $\mathcal{H} = \{ f \in \mathcal{F} : |f(z_0)| \geq 1 \}$

so $\mathcal{F} = \mathcal{G} \cup \mathcal{H}$

We now show that $\mathcal{G}$ is normal in $H(G)$ and that $\mathcal{H}$ is normal in $C(G, \mathbb{C}_\infty)$

1) <u>To prove $\mathcal{G}$ is normal in $H(G)$</u>

Let $a \in G$ be arbitrary and let γ be a curve in G from z_0 to a.

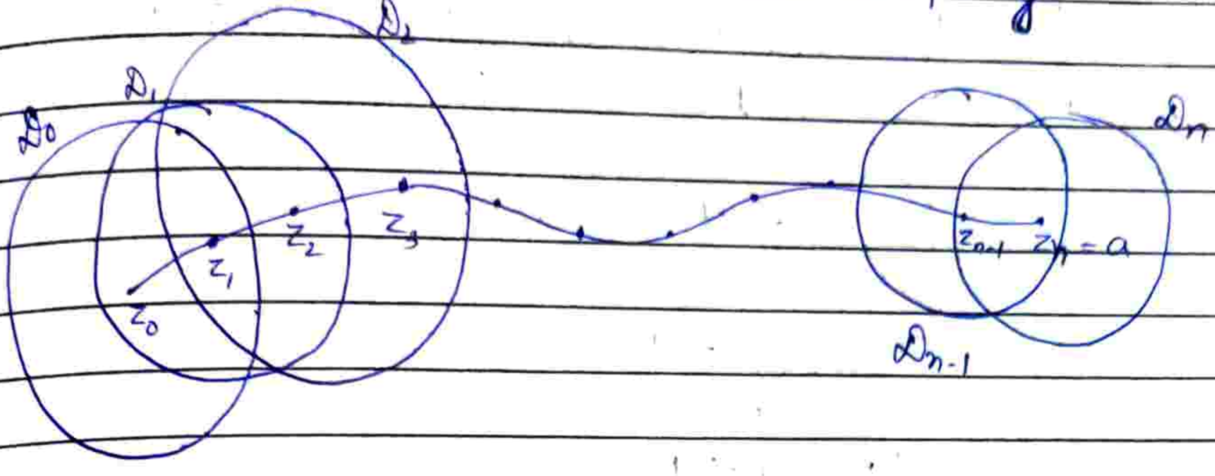
Suppose $D_0, D_1, \ldots, D_n$ be discs in G with centre $z_0, z_1, \ldots, z_n = a$ respectively on $\{\gamma\}$.

The discs are so constructed that z_{k-1} and z_k are in $D_{k-1} \cap D_k$ for $1 \leq k \leq n$

Assume that $\bar{D}_k \subset G$, for $1 \leq k \leq n$.

Now Applying Schottky's theorem to D_0, there is a constant C_0 such that

$$|f(z)| \leq C_0 \quad \forall\ z \in \mathcal{D}_0 \text{ and } f \in \mathcal{G}$$



We note that if $\mathcal{D}_0 = B(z_0; r)$ and $R > r$ is such that $\overline{B}(z_0; R) \subset G$ then we have

$$|f(z)| \leq C(1; B) \quad \forall\ z \in \mathcal{D}_0 \text{ and } f \in \mathcal{G}$$

whenever $r < BR$ $\{\because |f(0)| \leq 1\}$

Similarly for $\mathcal{D}_1$, there is a constant C_1 such that ~~$\overline{B}(z_0; R) \in G$ then we have~~

$$|f(z)| \leq C_1 \quad \forall\ z \in \mathcal{D}_1 \text{ and } f \in \mathcal{G}$$

so that $\mathcal{G}$ is uniformly bounded by a constant C_1 on $\mathcal{D}_1$

Continuing this process we have that $\mathcal{G}$ is uniformly bounded on $\mathcal{D}_n$

Since $a \in G$ was arbitrary, therefore $\mathcal{G}$ is locally bounded. Hence by Montel's Theorem, $\mathcal{G}$ is normal in $H(G)$

11) # To prove that $\mathcal{H}$ is normal in $C(G, \mathbb{C}_\infty)$

Let $f \in \mathcal{H}$

Then, $1/f$ is analytic on G because f never vanishes

Also $1/f$ never assume the value 1

i.e. $\frac{1}{f(z)} \neq 1 \quad \forall z$

$\Rightarrow \left|\frac{1}{f(z)}\right| \neq 1 \quad \forall z$

$\Rightarrow \left|\frac{1}{f}(z)\right| \neq 1 \quad \forall z$

$\Rightarrow \left|\frac{1}{f}(z_0)\right| \neq 1$

$\Rightarrow \left|\frac{1}{f}(z_0)\right| < 1 \qquad \left\{\begin{array}{l} \because f(z_0) \geq 1 \text{ in } \mathcal{H} \\ \therefore \frac{1}{f(z_0)} \leq 1 \text{ in } \mathcal{H} \end{array}\right\}$

If we define $\overline{\mathcal{H}} = \left\{ \frac{1}{f} : f \in \mathcal{H} \right\}$

Then $\overline{\mathcal{H}} \subseteq \mathcal{G}$

$\because \mathcal{G}$ is normal in $H(G)$

$\therefore \overline{\mathcal{H}}$ is also normal in $H(G)$

Thus if $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{H}$

then there is a subsequence $\{f_{n_k}\}_{n=1}^{\infty}$ and an analytic function h defined in G such that $\frac{1}{f_{n_k}} \to h$ in $H(G)$.

Hence by Hurwitz's theorem either $h \equiv 0$ or h never vanishes.

If $h \equiv 0$, then $f_{n_k}(z) \longrightarrow \infty$ uniformly on compact subsets of G

If h never vanishes, then $\frac{1}{h}$ is analytic and so $f_{n_k}(z) \to \frac{1}{h(z)}$ uniformly on compact subsets of G.

Normal :- A set $\mathcal{F} \subset C(G, \Omega)$ is said to be normal if each sequence in $\mathcal{F}$ has a subsequence which converges to a function f in $C(G, \Omega)$

Schottky's Theorem :-

Statement :- For each α and β, $0 < \alpha < \infty$ and $0 \leq \beta \leq 1$, there is a constant $c(\alpha, \beta)$ such that if f is an analytic function defined in some simply connected region containing $\bar{B}(0;1)$ that omits the values 0 and 1 and such that $|f(0)| \leq \alpha$; then

$$|f(z)| \leq c(\alpha, \beta) \quad \text{for } |z| \leq \beta$$

Proof :- Let G be a simply connected region containing the disc $\bar{B}(0,1)$ and suppose that f is a function defined in G and that f never assumes the values 0 and 1.

Suppose l is any branch of $\log f$

i.e. $l(z) = \log f(z)$

or $f(z) = e^{l(z)}$ —— ①

and define $F = \frac{l}{2\pi i}$ —— ②

$H = \sqrt{F} - \sqrt{F-1}$ —— ③

$$g = \text{a branch of } \log H$$

i.e. $g(z) = \log H(z)$

or $H(z) = e^{g(z)}$ —— ④

and specify these branches by requiring

$$0 \leq \operatorname{Im} l(0) < 2\pi$$

and $0 \leq \operatorname{Im} g(0) < 2\pi$

We now consider the following two cases:

<u>Case I</u>:- Suppose $\frac{1}{2} \leq |f(0)| \leq \alpha$

In this case we have,

$$F(z) = \frac{l(z)}{2\pi i}$$

$$\Rightarrow F(z) = \frac{\log f(z)}{2\pi i}$$

$$\Rightarrow |F(z)| = \left|\frac{\log f(z)}{2\pi i}\right| = \frac{|\log f(z)|}{2\pi}$$

$$\Rightarrow |F(z)| = \frac{1}{2\pi}\left|\log|f(z)| + i \operatorname{Im} \log f(z)\right| \quad \text{—— ⑤}$$

<u>Note</u>:- $|x+iy| = (x^2+y^2)^{1/2}$

and $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

$$\Rightarrow \log(x+iy) = \log(x^2+y^2)^{1/2} + i\tan^{-1}\frac{y}{x}$$

$$\Rightarrow \log(x+iy) = \log|x+iy| + i \operatorname{Im} \log(x+iy)$$

At $z=0$,

$$|F(0)| = \frac{1}{2\pi}\left|\log|f(0)| + i \operatorname{Im} \log f(0)\right|$$

$$\Rightarrow |F(0)| \leq \frac{1}{2\pi} \left| \log |f(0)| \right| + \frac{1}{2\pi} |i| \left| \text{Im} \log f(0) \right|$$

$$\Rightarrow |F(0)| \leq \frac{1}{2\pi} \left| \log |f(0)| \right| + \frac{1}{2\pi} \left| \text{Im}\, l(0) \right|$$

$$\Rightarrow |F(0)| \leq \frac{1}{2\pi} |\log \alpha| + 1 = C_0(\alpha) \quad \text{(say)}$$

$$\Rightarrow |F(0)| \leq C_0(\alpha) \quad \text{—— (6)}$$

Also we see that,

$$\left| \sqrt{F(0)} \pm \sqrt{F(0)-1} \right| \leq \left| \sqrt{F(0)} \right| + \left| \sqrt{F(0)-1} \right|$$

$$= |F(0)|^{1/2} + |F(0)-1|^{1/2}$$

$$\leq [C_0(\alpha)]^{1/2} + [|F(0)| + 1]^{1/2}$$

$$\leq [C_0(\alpha)]^{1/2} + [C_0(\alpha) + 1]^{1/2}$$

$$= C_1(\alpha) \quad \text{(say)}$$

$$\Rightarrow \left| \sqrt{F(0)} \pm \sqrt{F(0)-1} \right| \leq C_1(\alpha) \quad \text{—— (7)}$$

Also, $|g(0)| = |\log H(0)|$

$$= \left| \log |H(0)| + i \,\text{Im} \log H(0) \right|$$

$$= \left| \log |H(0)| + i \,\text{Im}\, g(0) \right|$$

$$\Rightarrow |g(0)| = \left| \log |H(0)| + i \,\text{Im}\, g(0) \right|$$

$$\leq \left| \log |H(0)| \right| + |i| \left| \text{Im}\, g(0) \right|$$

$$\leq \left| \log |H(0)| \right| + |2\pi|$$

$$= \left| \log \left| \sqrt{F(0)} - \sqrt{F(0)-1} \right| \right| + 2\pi$$

$$\leq |\log C_1(\alpha)| + 2\pi = C_2(\alpha) \quad \text{(say)}$$

$$\Rightarrow \boxed{|g(0)| \leq C_2(\alpha)} \quad \text{—— (8)}$$

Now, if $|z_0| < 1$, then the region $g(B(z_0, 1-|z_0|))$ contains a disc of radius $(1-|z|)\, L\, |g'(z_0)|$

Also we know that $g(B(0,1))$ contains no disc of radius 1

Thus $(1-|z_0|)\, L\, |g'(z_0)| < 1$

$$\Rightarrow \quad |g'(z_0)| < \frac{1}{L(1-|z_0|)} \qquad \text{— (9)}$$

Now, let $|z_0| < 1$ and γ be the line segment $[0, z_0]$

Then,

$$
\begin{aligned}
|g(z_0)| &= |g(z_0) + g(0) - g(0)| \\
&\le |g(z_0) - g(0)| + |g(0)| \\
&\le \left| \int_0^{z_0} g'(z)\, dz \right| + C_2(\alpha) \\
&= \left| \int_\gamma g'(z)\, dz \right| + C_2(\alpha) \\
&\le \max_{z \in [0, z_0]} |g'(z)| \int_\gamma |dz| + C_2(\alpha) \\
&= C_2(\alpha) + |z_0| \max_{z \in [0, z_0]} |g'(z)|
\end{aligned}
$$

$$\Rightarrow |g(z_0)| \le C_2(\alpha) + \frac{|z_0|}{L(1-|z_0|)} \cancel{\max_{z\in[0,z_0]} |g'(z)|} \quad \{\text{from (9)}\}$$

If $C_3(\alpha, \beta) = C_2(\alpha) + \beta\, [L(1-\beta)]^{-1}$

Then we get

$$\boxed{|g(z)| \le C_3(\alpha, \beta)} \text{ for } |z| \le \beta \qquad \text{— (10)}$$

Again if $|z| \le \beta$ then

$$|f(z)| = \left|\exp\left[\pi i \cosh 2g(z)\right]\right|$$

$$\Rightarrow |f(z)| \le \exp\left\{\left|\pi i \cosh 2g(z)\right|\right\}$$

$$\Rightarrow |f(z)| \le \exp\left\{\pi \left|\cosh 2g(z)\right|\right\}$$

$$\Rightarrow |f(z)| \le \exp\left\{\pi \left|\frac{e^{2g(z)} + e^{-2g(z)}}{2}\right|\right\}$$

$$\Rightarrow |f(z)| \le \exp\left[\frac{\pi}{2}\left\{\left|e^{2g(z)}\right| + \left|e^{-2g(z)}\right|\right\}\right]$$

$$\Rightarrow |f(z)| \le \exp\left\{\frac{\pi}{2}\left\{\left|e^{2g(z)}\right| + \left|e^{2g(z)}\right|\right\}\right\} \quad (\because e^{-x} \le e^{x})$$

$$\Rightarrow |f(z)| \le \exp\left\{\pi \left|e^{2g(z)}\right|\right\}$$

$$\Rightarrow |f(z)| \le \exp\left\{\pi e^{2|g(z)|}\right\}$$

$$\Rightarrow |f(z)| \le \exp\left\{\pi e^{2C_3(\alpha,\beta)}\right\} = C_4(\alpha,\beta) \text{ (say)} \quad (\text{By } 10)$$

$$\Rightarrow \boxed{|f(z)| \le C_4(\alpha,\beta)} \quad \text{—— } (11)$$

Case II :- Suppose $0 < |f(0)| < \frac{1}{2}$

In this case, $(1-f)$ satisfies the conditions of cases so that

$$\boxed{|(1-f)(z)| \le C_4(\alpha,\beta)} \quad \text{—— } (12)$$

Thus. $|f(z)| = |1 - (1 - f(z))|$

$$\Rightarrow |f(z)| \le 1 + |1 - f(z)|$$

$$\Rightarrow |f(z)| \le 1 + C_4(\alpha,\beta) \quad (\text{By } 12)$$

$$\Rightarrow |f(z)| \le C_5(\alpha,\beta) \quad \text{(say)}$$

Let $c(\alpha, \beta) = \max \{ c_4(\alpha, \beta), c_5(\alpha, \beta) \}$.

Then from case ① and case ②, we conclude that

$$\boxed{|f(z)| \leq c(\alpha, \beta)}$$

This completes the proof.

1/4 – Theorem

Theorem:- Let $f \in \mathcal{J}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

Then (a) $|a_2| \leq 2$

(b) $f(U) \supset \mathcal{D}\left(0; \frac{1}{4}\right)$ i.e. $f(U)$ contains all w with $|w| < \frac{1}{4}$.

Proof:- (a) We have the result:

"If $f \in \mathcal{J}$, there exists a $g \in \mathcal{J}$ such that $g^2(z) = f(z^2) \ \forall \ z \in U$".

Let $G(z) = \dfrac{1}{g(z)}$

then $G(z) = \dfrac{1}{\sqrt{f(z^2)}}$

$$= \frac{1}{\sqrt{z^2 + \sum_{n=2}^{\infty} a_n z^{2n}}}$$

$$= \frac{1}{z\sqrt{1 + \sum_{n=2}^{\infty} a_n z^{2n-2}}}$$

$$= \frac{1}{z}\left(1 + \sum_{n=2}^{\infty} a_n z^{2n-2}\right)^{-1/2}$$

$$= \frac{1}{z}\left[1 - \frac{1}{2}\sum_{n=2}^{\infty} a_n z^{2n-2} + \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)-1\right)}{\lfloor 2}\left(\sum_{n=2}^{\infty} a_n z^{2n-2}\right)^2 \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{2}\sum_{n=2}^{\infty} a_n z^{2n-2} + \frac{3}{8}\left(\sum_{n=2}^{\infty} a_n z^{2n-2}\right)^2 - \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{a_2}{2} z^2 + \cdots\right]$$

$$= \frac{1}{z} - \frac{a^2 z}{2} + \cdots$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} \alpha_n z^n \quad (\text{say}) \ \forall z \in U$$

Again we know the result:

"If $F \in H(U - \{0\})$, F is one-one in U and $F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \alpha_n z^n$, then $|\alpha_1| \leq 1$."

Hence $|\alpha_1| \leq 1$

$$\Rightarrow \left|\frac{-a_2}{2}\right| = \frac{|a_2|}{2} \leq 1$$

$$\Rightarrow |a_2| \leq 2$$

Proved (a).

(b) Suppose $w \notin f(U)$

Define $h(z) = \dfrac{f(z)}{1 - f(z)/w}$

Since $f \in H(U)$, therefore $h \in H(U)$

To prove h is one to one

Let $z_1, z_2 \in U$

then $h(z_1) = h(z_2) \Rightarrow \dfrac{f(z_1)}{1 - f(z_1)/w} = \dfrac{f(z_2)}{1 - f(z_2)/w}$

$$\Rightarrow f(z_1)\left(1-\frac{f(z_2)}{w}\right) = f(z_2)\left(1-\frac{f(z_1)}{w}\right)$$

$$\Rightarrow f(z_1) - \frac{1}{w} f(z_1)f(z_2) = f(z_2) - \frac{1}{w} f(z_1)f(z_2)$$

$$\Rightarrow f(z_1) = f(z_2)$$

$$\Rightarrow z_1 = z_2 \qquad \{\because f \text{ is one-one}\}$$

$\Rightarrow h$ is one to one

Also, $$h(z) = \frac{f(z)}{1 - f(z)/w}$$

$$= f(z)\left[1-\frac{f(z)}{w}\right]^{-1}$$

$$= \left(z+\sum_{n=2}^{\infty} a_n z^n\right)\left[1+\frac{f(z)}{w}+\cdots\right]$$

$$= \left(z+\sum_{n=2}^{\infty} a_n z^n\right)\left(1+\frac{z}{w}+\cdots\right)$$

$$\left\{\because f(z) = z+\sum a_n z^n\right\}$$

$$= \left(z+a_2z^2+\cdots\right)\left(1+\frac{z}{w}+\cdots\right)$$

$$\Rightarrow h(z) = z+\left(a_2+\frac{1}{w}\right)z^2+\cdots$$

$$\Rightarrow h'(z) = 1+2\left(a_2+\frac{1}{w}\right)z+\cdots$$

Hence $h(0)=0$ and $h'(0)=1$

Thus, $h \in \mathcal{J}$

Finally applying (a) to h, then we get

$$\left|\frac{1}{w}+a_2\right| \le 2$$

or $$\left|\frac{1}{w}\right| - |a_2| \le \left|\frac{1}{w}+a_2\right| \le 2$$

$$\Rightarrow \left|\frac{1}{w}\right| \leq |a_2| + 2$$

$$\Rightarrow \left|\frac{1}{w}\right| \leq 4 \quad \cancel{\text{for every } w \notin f(U)}$$

$$\Rightarrow |w| \geq \frac{1}{4} \quad \text{for every } w \notin f(U)$$

$$\Rightarrow f(U) \supset \mathcal{D}\left(0; \frac{1}{4}\right)$$

Proved (b).